PLANE AND SOLID GEOMETRY

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WENTWORTH-SMITH MATHEMATICAL SERIES

PLANE AND SOLID GEOMETRY

BY

GEORGE WENTWORTH

AND

DAVID EUGENE SMITH

GINN AND COMPANY
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PREFACE

Long after the death of Robert Recorde, England's first great writer of textbooks, the preface of a new edition of one of his works contained the appreciative statement that the book was "entail'd upon the People, ratified and sign'd by the approbation of Time." The language of this sentiment sounds quaint, but the noble tribute is as impressive to-day as when first put in print two hundred and fifty years ago.

With equal truth these words may be applied to the Geometry written by George A. Wentworth. For a generation it has been the leading textbook on the subject in America. It set a standard for usability that every subsequent writer upon geometry has tried to follow, and the number of pupils who have testified to its excellence has run well into the millions.

In undertaking to prepare a work to take the place of the Wentworth Geometry the authors have been guided by certain well-defined principles, based upon an extended investigation of the needs of the schools and upon a study of all that is best in the recent literature of the subject. The effects of these principles they feel should be summarized for the purpose of calling the attention of the wide circle of friends of the Wentworth-Smith series to the points of similarity and of difference in the two works.

1. Every effort has been made not only to preserve but to improve upon the simplicity of treatment, the clearness of expression, and the symmetry of page that characterized the successive editions of the Wentworth Geometry. It has been the purpose to prepare a book that should do even more than maintain the traditions this work has fostered.

2. The proofs have been given substantially in full, to the end that the pupil may always have before him a model for his independent treatment of the exercises.

3. The sequence of propositions has been improved in several respects, notably in the treatment of parallels.

4. To meet a general demand, the number of propositions has been decreased so as to include only the great basal theorems and problems. A little of the less important material has been placed in the Appendixes, to be used or not as circumstances demand.

5. The exercises, in some respects the most important part of a course in geometry, have been rendered more dignified in appearance and have been improved in content. The number of simple exercises has been greatly increased, while the difficult puzzle is much less in evidence than in most American textbooks. The exercises are systematically grouped, appearing in full pages, in large type, at frequent intervals. They are not all intended for one class, but are so numerous as to allow the teacher to make selections from year to year.

6. The work throughout has been made as concrete as is reasonable. Definitions have been postponed until they are actually needed, only well-recognized terms have been employed, the pupil is initiated at once into the practical use of the instruments, some of the reasons for studying geometry are early shown in an interesting way, application of geometry to practical cases in mensuration is provided for, and correlation is made with the algebra already studied.

The authors are indebted to many friends of the Wentworth-Smith series for assistance and encouragement in the labor of preparing this work, and they will welcome any further suggestions for improvement from any of their readers.

GEORGE WENTWORTH
DAVID EUGENE SMITH

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SYMBOLS AND ABBREVIATIONS

=	equals, equal, equal to, is equal to, or is equivalent to.
>	is greater than.
<	is less than.
∥	parallel.
⊥	perpendicular.
∠	angle.
△	triangle.
▱	parallelogram.
▭	rectangle.
⊙	circle.
st.	straight.
rt.	right.
∵	since.
∴	therefore.

Adj.	adjacent.
Alt.	alternate.
Ax.	axiom.
Const.	construction.
Cor.	corollary.
Def.	definition.
Ex.	exercise.
Ext.	exterior.
Fig.	figure.
Hyp.	hypothesis.
Iden.	identity.
Int.	interior.
Post.	postulate.
Prob.	problem.
Prop.	proposition.
Sup.	supplementary.

These symbols take the plural form when necessary, as in the case of ∥s, ∠s, △s, ⊙s.

The symbols +, −, ×, ÷ are used as in algebra.

There is no generally accepted symbol for "is congruent to," and the words are used in this book. Some teachers use ≙ or ≅, and some use ≡, but the sign of equality is more commonly employed, the context telling whether equality, equivalence, or congruence is to be understood.

Q. E. D. is an abbreviation that has long been used in geometry for the Latin words *quod erat demonstrandum*, "which was to be proved."

Q. E. F. stands for *quod erat faciendum*, "which was to be done."

PLANE GEOMETRY

INTRODUCTION

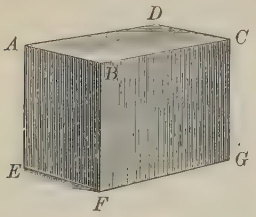
計算

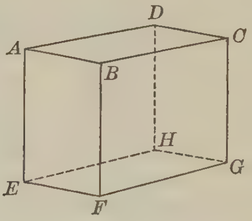
1. The Nature of Arithmetic. In arithmetic we study computation, the working with numbers. We may have a formula expressed in algebraic symbols, such as $a = bh$, where a may stand for the area of a rectangle, and b and h respectively for the number of units of length in the base and height; but the actual computation involved in applying such formula to a particular case is part of arithmetic.

2. The Nature of Algebra. In algebra we generalize the arithmetic, and instead of saying that the area of a rectangle with base 4 in. and height 2 in. is 4×2 sq. in., we express a general law by saying that $a = bh$. In arithmetic we may have an equality, like $2 \times 16 + 17 = 49$, but in algebra we make much use of equations, like $2x + 17 = 49$. Algebra, therefore, is a generalized arithmetic.

3. The Nature of Geometry. We are now about to begin another branch of mathematics, one not chiefly relating to numbers although it uses numbers, and not primarily devoted to equations although using them, but one that is concerned principally with the study of forms, such as triangles, parallelograms, and circles. Many facts that are stated in arithmetic and algebra are proved in geometry. For example, in geometry it is proved that the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides, and that the circumference of a circle equals 3.1416 times the diameter.

4. Solid. The block here represented is called a *solid;* it is a limited portion of space filled with matter. In geometry, however, we have nothing to do with the matter of which a





body is composed; we study simply its *shape* and *size*, as in the second figure.

That is, a physical solid can be touched and handled; a geometric solid is the space that a physical solid is conceived to occupy. For example, a stick is a physical solid; but if we put it into wet plaster, and then remove it, the hole that is left may be thought of as a geometric solid although it is filled with air.

5. Geometric Solid. A limited portion of space is called a *geometric solid.*

6. Dimensions. The block represented in § 4 extends in three principal directions:

(1) From left to right, that is, from A to D;

(2) From back to front, that is, from A to B;

(3) From top to bottom, that is, from A to E.

These extensions are called the *dimensions* of the block, and are named in the order given, *length, breadth* (or width), and *thickness* (height, altitude, or depth). Similarly, we may say that every solid has three dimensions.

Very often a solid is of such shape that we cannot point out the length, or distinguish it from the breadth or thickness, as an irregular block of coal. In the case of a round ball, where the length, breadth, and thickness are all the same in extent, it is impossible to distinguish one dimension from the others.

7. Surface. The block shown in § 4 has six flat faces, each of which is called a *surface*. If the faces are made smooth by polishing, so that when a straight edge is applied to any one of them the straight edge in every part will touch the surface, each face is called a *plane surface*, or a *plane*.

These surfaces are simply the boundaries of the solid. They have no thickness, even as a colored light shining upon a piece of paper does not make the paper thicker. A board may be planed thinner and thinner, and then sandpapered still thinner, thus coming nearer and nearer to representing what we think of as a geometric plane, but it is always a solid bounded by surfaces.

That which has length and breadth without thickness is called a *surface*.

8. Line. In the solid shown in § 4 we see that two adjacent surfaces intersect in a line. A line is therefore simply the boundary of a surface, and has neither breadth nor thickness.

That which has length without breadth or thickness is called a *line*.

A telegraph wire, for example, is not a line. It is a solid. Even a pencil mark has width and a very little thickness, so that it is also a solid. But if we think of a wire as drawn out so that it becomes finer and finer, it comes nearer and nearer to representing what we think of and speak of as a geometric line.

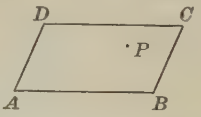
9. Magnitudes. Solids, surfaces, and lines are called *magnitudes*.

10. Point. In the solid shown in § 4 we see that when two lines meet they meet in a point. A point is therefore simply the boundary of a line, and has no length, no breadth, and no thickness.

That which has only position, without length, breadth, or thickness, is called a *point*.

We may think of the extremity of a line as a point. We may also think of the intersection of two lines as a point, and of the intersection of two surfaces as a line.

11. Representing Points and Geometric Magnitudes. Although we only imagine such geometric magnitudes as lines or planes, we may represent them by pictures.



Thus we represent a point by a fine dot, and name it by a letter, as P in this figure.

We represent a line by a fine mark, and name it by letters placed at the ends, as AB.

We represent a surface by its boundary lines, and name it by letters placed at the corners or in some other convenient way, as $ABCD$.

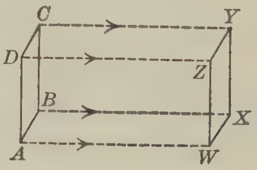
We represent a solid by the boundary faces or by the lines bounding the faces, as in § 4.

12. Generation of Geometric Magnitudes. We may think of

(1) A line as generated by a moving point;
(2) A surface as generated by a moving line;
(3) A solid as generated by a moving surface.

For example, as shown in the figure let the surface $ABCD$ move to the position $WXYZ$. Then

(1) A generates the line AW;
(2) AB generates the surface $AWXB$;
(3) $ABCD$ generates the solid AY.



Of course a point will not generate a line by simply turning over, for this is not motion for a point; nor will a line generate a surface by simply sliding along itself; nor will a surface generate a solid by simply sliding upon itself.

13. Geometric Figure. A point, a line, a surface, a solid, or any combination of these, is called a *geometric figure*.

A geometric figure is generally called simply a *figure*.

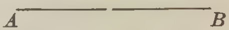
14. Geometry. The science of geometric figures is called *geometry*.

Plane geometry treats of figures that lie wholly in the same plane, that is, of plane figures.

Solid geometry treats of figures that do not lie wholly in the same plane.

15. Straight Line. A line such that any part placed with its ends on any other part must lie wholly in the line is called a *straight line.*

For example, AB is a straight line, for if we take, say, a half inch of it, and place it in any way on any other part of AB, but so that its ends lie in AB, then the whole of the half inch of line will lie in AB. This is well shown by using tracing paper. The word *line* used alone is understood to mean a straight line.

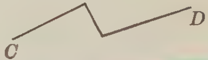


Part of a straight line is called a *segment* of the line. The term *segment* is applied also to certain other magnitudes.

16. Equality of Lines. Two straight-line segments that can be placed one upon the other so that their extremities coincide are said to be *equal.*

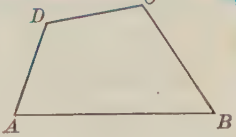
In general, two geometric magnitudes are equal if they can be made to coincide throughout their whole extent. We shall see later that some figures that coincide are said to be *congruent.*

17. Broken Line. A line made up of two or more different straight lines is called a *broken line.*



For example, CD is a broken line.

18. Rectilinear Figure. A plane figure formed by a broken line is called a *rectilinear figure.*



For example, $ABCD$ is a rectilinear figure.

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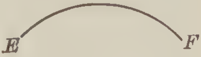
19. Curve Line. A line no part of which is straight is called a *curve line,* or simply a *curve.*

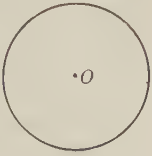


For example, EF is a curve line.

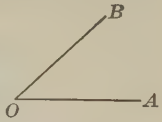
20. Curvilinear Figure. A plane figure formed by a curve line is called a *curvilinear figure.*

For example, O is a curvilinear figure with which we are already familiar.

Some curvilinear figures are surfaces bounded by curves and others are the curves themselves.

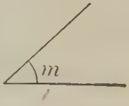


21. Angle. The opening between two straight lines drawn from the same point is called an *angle.*



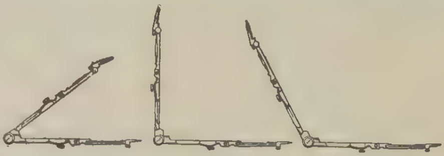
Strictly speaking, this is a *plane* angle. We shall find later that there are angles made by curve lines and angles made by planes.

The two lines are called the *sides* of the angle, and the point of meeting is called the *vertex.*



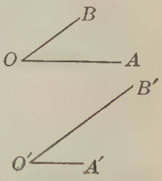
An angle may be read by naming the letters designating the sides, the vertex letter being between the others, as the angle AOB. An angle may also be designated by the vertex letter, as the angle O, or by a small letter within, as the angle m. A curve is often drawn to show the particular angle meant, as in angle m.

22. Size of Angle. The size of an angle depends upon the amount of turning necessary to bring one side into the position of the other.



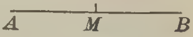
One angle is greater than another angle when the amount of turning is greater. Thus in these compasses the first angle is smaller than the second, which is also smaller than the third. The length of the sides has nothing to do with the size of the angle.

23. Equality of Angles. Two angles that can be placed one upon the other so that their vertices coincide and the sides of one lie along the sides of the other are said to be *equal.*



For example, the angles AOB and $A'O'B'$ (read "A prime, O prime, B prime") are equal. It is well to illustrate this by tracing one on thin paper and placing it upon the other.

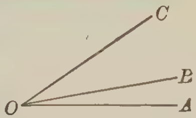
24. Bisector. A point, a line, or a plane that divides a geometric magnitude into two equal parts is called a *bisector* of the magnitude.



For example, M, the mid-point of the line AB, is a bisector of the line.

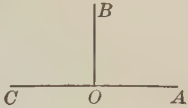
25. Adjacent Angles. Two angles that have the same vertex and a common side between them are called *adjacent angles.*

For example, the angles AOB and BOC are adjacent angles, and in § 26 the angles AOB and BOC are adjacent angles.



26. Right Angle. When one straight line meets another straight line and makes the adjacent angles equal, each angle is called a *right angle.*

For example, angles AOB and BOC in this figure. If CO is cut off, angle AOB is still a right angle.

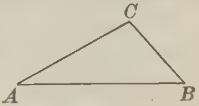


27. Perpendicular. A straight line making a right angle with another straight line is said to be *perpendicular* to it.

Thus OB is perpendicular to CA, and CA to OB. OB is also called *a perpendicular* to CA, and O is called the *foot* of the perpendicular OB.

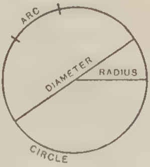
28. Triangle. A portion of a plane bounded by three straight lines is called a *triangle.*

The lines AB, BC, and CA are called the *sides* of the triangle ABC, and the sides taken together form the *perimeter.* The points A, B, and C are the *vertices* of the triangle, and the angles A, B, and C are the *angles* of the triangle. The side AB upon which the triangle is supposed to rest is the *base* of the triangle. Similarly for other plane figures.



29. Circle. A closed curve lying in a plane, and such that all of its points are equally distant from a fixed point in the plane, is called a *circle.*

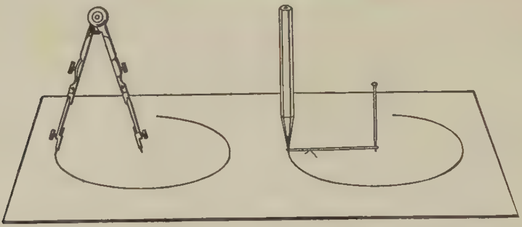
The length of the circle is called the *circumference.* The point from which all points on the circle are equally distant is the *center.* Any portion of a circle is an *arc.* A straight line from the center to the circle is a *radius.* A straight line through the center, terminated at each end by the circle, is a *diameter.*



Formerly in elementary geometry *circle* was taken to mean the space inclosed, and the bounding line was called the circumference. Modern usage has conformed to the definition used in higher mathematics.

30. Instruments of Geometry. In geometry only two instruments are necessary besides pencil and paper. These are a straight edge, or ruler, and a pair of compasses.

It is evident that *all radii of the same circle are equal.*



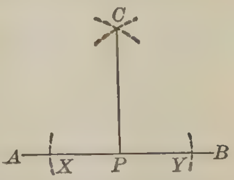
In the absence of compasses, and particularly for blackboard work, a loop made of string may be used. For the accurate transfer of lengths, however, compasses are desirable.

31. Exercises in using Instruments. The following simple exercises are designed to accustom the pupil to the use of instruments. No proofs are attempted, these coming later in the course.

This section may be omitted if desired, without affecting the course.

EXERCISE 1

1. From a given point on a given straight line required to draw a perpendicular to the line.



Let AB be the given line and P be the given point.

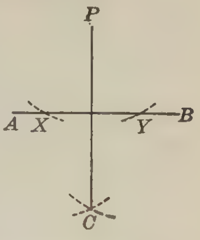
It is required to draw from P a line perpendicular to AB.

With P as a center and any convenient radius draw arcs cutting AB at X and Y.

With X as a center and XY as a radius draw a circle, and with Y as a center and the same radius draw another circle, and call one intersection of the circles C.

With a straight edge draw a line from P to C, and this will be the perpendicular required.

2. From a given point outside a given straight line required to let fall a perpendicular to the line.



Let AB be the given straight line and P be the given point.

It is required to draw from P a line perpendicular to AB.

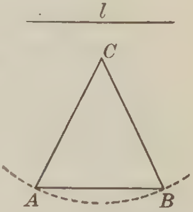
With P as a center and any convenient radius draw an arc cutting AB at X and Y.

With X as a center and any convenient radius draw a circle, and with Y as a center and the same radius draw another circle, and call one intersection of the circles C.

With a straight edge draw a straight line from P to C, and this will be the perpendicular required.

It is interesting to test the results in Exs. 1 and 2, by cutting the paper and fitting the angles together.

3. Required to draw a triangle having two sides each equal to a given line.



Let l be the given line.

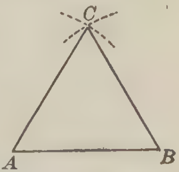
It is required to draw a triangle having two sides each equal to l.

With any center, as C, and a radius equal to l draw an arc.

Join any two points on the arc, as A and B, with each other and with C by straight lines.

Then ABC is the triangle required.

4. Required to draw a triangle having its three sides each equal to a given line.



Let AB be the given line.

It is required to draw a triangle having its three sides each equal to AB.

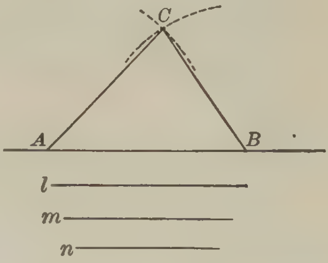
With A as a center and AB as a radius draw a circle, and with B as a center and the same radius draw another circle.

Join either intersection of the circles with A and B by straight lines.

Then ABC is the triangle required.

In such cases draw the arcs only long enough to show the point of intersection.

5. Required to draw a triangle having its sides equal respectively to three given lines.



Let the three lines be *l*, *m*, and *n*.

What is now required?

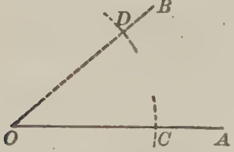
Upon any line mark off with the compasses a line-segment *AB* equal to *l*.

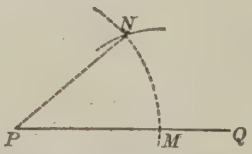
With *A* as a center and *m* as a radius draw a circle; with *B* as a center and *n* as a radius draw a circle.

Draw *AC* and *BC*.

Then *ABC* is the required triangle.

6. From a given point on a given line required to draw a line making an angle equal to a given angle.





Let *P* be the given point on the given line *PQ*, and let angle *AOB* be the given angle.

What is now required?

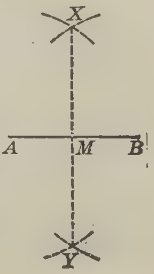
With *O* as a center and any radius draw an arc cutting *AO* at *C* and *BO* at *D*.

With *P* as a center and *OC* as a radius draw an arc cutting *PQ* at *M*.

With *M* as a center and the straight line joining *C* and *D* as a radius draw an arc cutting the arc just drawn at *N*, and draw *PN*.

Then angle *MPN* is the required angle.

7. Required to bisect a given straight line



Let *AB* be the given line.

It is required to bisect *AB*.

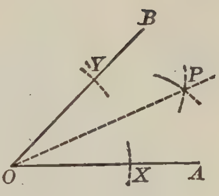
With *A* as a center and *AB* as a radius draw a circle, and with *B* as a center and the same radius draw a circle.

Call the two intersections of the circles *X* and *Y*.

Draw the straight line *XY*.

Then *XY* bisects the line *AB* at the point of intersection *M*.

8. Required to bisect a given angle.



Let AOB be the given angle.

It is required to bisect the angle AOB.

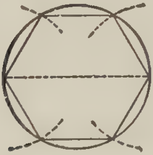
With O as a center and any convenient radius draw an arc cutting OA at X and OB at Y.

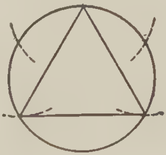
With X as a center and a line joining X and Y as a radius draw a circle, and with Y as a center and the same radius draw a circle, and call one point of intersection of the circles P.

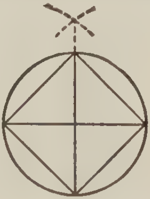
Draw the straight line OP.

Then OP is the required bisector.

9. By the use of compasses and ruler draw the following figures:

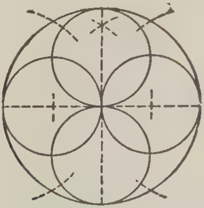


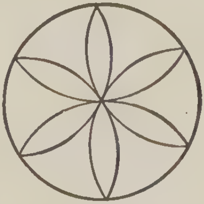




The dotted lines show how to fix the points needed in drawing the figure, and they may be erased after the figure is completed. In general, in geometry, auxiliary lines (those needed only as aids) are indicated by dotted lines.

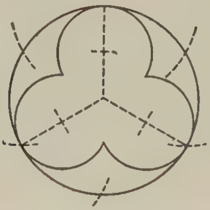
10. By the use of compasses and ruler draw the following figures:

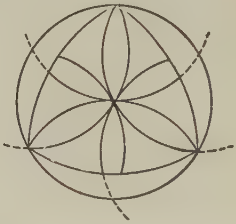




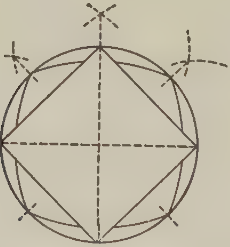
It is apparent from the figures in Exs. 9 and 10 that the radius of the circle may be used in describing arcs that shall divide the circle into six equal parts.

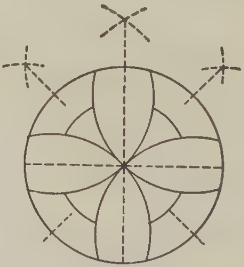
11. By the use of compasses and ruler draw the following figures:



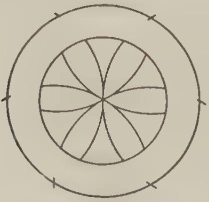


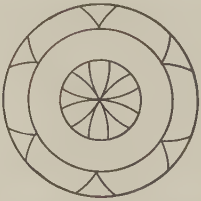
12. By the use of compasses and ruler draw the following figures:





13. By the use of compasses and ruler draw the following figures:





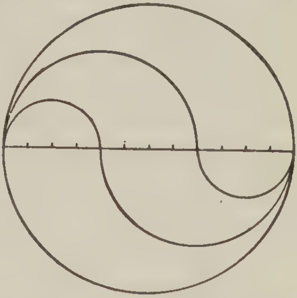
In such figures artistic patterns may be made by coloring various portions of the drawings. In this way designs are made for stained-glass windows, for oilcloth, for colored tiles, and for other decorations.

14. Draw a triangle of which each side is $1\frac{1}{4}$ in.

15. Draw two lines bisecting each other at right angles.

16. Bisect each of the four right angles formed by two lines bisecting each other at right angles.

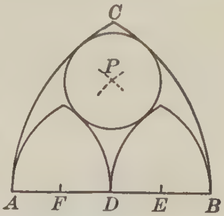
17. Draw a line $1\frac{1}{2}$ in. long and divide it into eighths of an inch, using the ruler. Then with the compasses draw this figure.



It is easily shown, when we come to the measurement of the circle, that these two curve lines divide the space inclosed by the circle into parts that are exactly equal to one another.

By continuing each semicircle to make a complete circle another interesting figure is formed. Other similar designs are easily invented, and students should be encouraged to make such original designs.

18. In planning a Gothic window this drawing is needed. The arc BC is drawn with A as a center and AB as a radius. The small arches are described with A, D, and B as centers and AD as a radius. The center P is found by taking A and B as centers and AE as a radius. How may the points D, E, and F be found? Draw the figure.



19. Draw a triangle of which each side is 1 in. Bisect each side, and with the points of bisection as centers and with radii $\frac{1}{2}$ in. long draw three circles.

20. A baseball diamond is a square 90 ft. on a side. Draw the plan, using a scale of $\frac{1}{16}$ in. to a foot. Locate the pitcher 60 ft. from the home plate.

21. A man travels from A directly east 1 mi. to B. He then turns and travels directly north $1\frac{3}{4}$ mi. to C. Draw the plan and find by measurement the distance AC to the nearest quarter of a mile. Use a scale of $\frac{1}{2}$ in. to a mile.

22. A double tennis court is 78 ft. long and 36 ft. wide. The net is placed 39 ft. from each end and the service lines 18 ft. from each end. Draw the plan, using a scale of $\frac{1}{16}$ in. to a foot, making the right angles as shown in Ex. 1. The accuracy of the construction may be tested by measuring the diagonals, which should be equal.

23. At the entrance to New York harbor is a gun having a range of 12 mi. Draw a line inclosing the range of fire, using a scale of $\frac{1}{16}$ in. to a mile

24. Two forts are placed on opposite sides of a harbor entrance, 13 mi. apart. Each has a gun having a range of 10 mi. Draw a plan showing the area exposed to the fire of both guns, using a scale of $\frac{1}{16}$ in. to a mile.

25. Two forts, A and B, are placed on opposite sides of a harbor entrance, 16 mi. apart. On an island in the harbor, 12 mi. from A and 11 mi. from B, is a fort C. The fort A has a gun with a range of 12 mi., fort B one with a range of 11 mi., and fort C one with a range of 10 mi. Draw a plan of the entrance to the harbor, showing the area exposed to the fire of each gun.

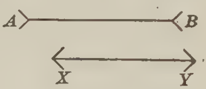
26. A horse, tied by a rope 25 ft. long at the corner of a lot 50 ft. square, grazes over as much of the lot as possible. The next day he is tied at the next corner, the third day at the third corner, and the fourth day at the fourth corner. Draw a plan showing the area over which he has grazed during the four days, using a scale of $\frac{1}{8}$ in. to 5 ft.

27. A gardener laid out a flower bed on the following plan: He made a triangle ABC, 16 ft. on a side, and then bisected two of the angles. From the point of intersection of the bisectors, P, he drew perpendiculars to the three sides of the triangle, PX, PY, and PZ. Then he drew a circle with P as a center and PX as a radius, and found that it just fitted in the triangle. Draw the plan, using a scale of $\frac{1}{4}$ in. to a foot.

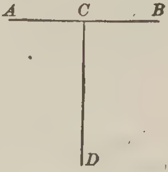
32. Necessity for Proof. Although part of geometry consists in drawing figures, this is not the most important part. It is essential to prove that the figures are what we claim them to be. The danger of trusting to appearances is seen in Exercise 2.

EXERCISE 2

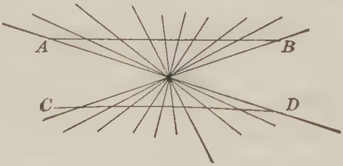
1. Estimate which is the longer line, AB or XY, and how much longer. Then test your estimate by measuring with the compasses or with a piece of paper carefully marked.



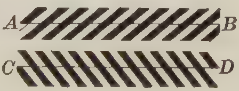
2. Estimate which is the longer line, AB or CD, and how much longer. Then test your estimate by measuring as in Ex. 1.



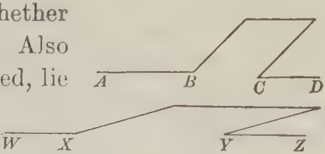
3. Look at this figure and state whether AB and CD are both straight lines. If one is not straight, which one is it? Test your answer by using a ruler or the folded edge of a piece of paper.



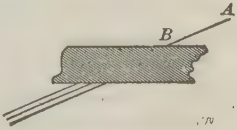
4. Look at this figure and state whether AB and CD are the same distance apart at A and C as at B and D. Then test your answer as in Ex. 1.



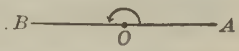
5. Look at this figure and state whether AB will, if prolonged, lie on CD. Also state whether WX will, if prolonged, lie on YZ. Then test your answer by laying a ruler along the lines.



6. Look at this figure and state which of the three lower lines is AB prolonged. Then test your answer by laying a ruler along AB.



33. Straight Angle. When the sides of an angle extend in opposite directions, so as to be in the same straight line, the angle is called a *straight angle.*

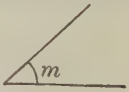


For example, the angle AOB, as shown in this figure, is a straight angle. The angle BOA, below the line, is also a straight angle.

34. Right Angle and Straight Angle. It follows from the definition of right angle (§ 26) that *a right angle is half of a straight angle.*

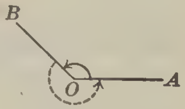
In like manner, it follows that *a straight angle equals twice a right angle.*

35. Acute Angle. An angle less than a right angle is called an *acute angle.*



For example, the angle m, as shown in this figure, is an acute angle

36. Obtuse Angle. An angle greater than a right angle and less than a straight angle is called an *obtuse angle.*



For example, the angle AOB, as shown in this figure, is an obtuse angle.

37. Reflex Angle. An angle greater than a straight angle and less than two straight angles is called a *reflex angle.*

For example, the angle BOA, marked with a dotted curve line in the figure in § 36, is a reflex angle.

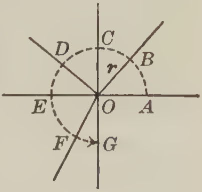
When we speak of an angle formed by two given lines drawn from a point we mean the smaller angle unless the contrary is stated.

38. Oblique Angles. Acute angles and obtuse angles are called *oblique angles.*

The sides of oblique angles are said to be *oblique* to each other, and are called *oblique lines.*

Evidently if we bisect a straight angle, we form two right angles; if we bisect a right angle or an obtuse angle, we form two acute angles; if we bisect a reflex angle, we form two obtuse angles.

39. Generation of Angles. Suppose the line r to revolve from the position OA about the point O as a vertex to the position OB. Then r describes or generates the *acute angle* AOB, and, as we have seen (§ 22) *the size of the angle depends upon the amount of rotation,* the angle being greater as the amount of turning is greater.



If r rotates still further, to the position OC, it has then generated the *right angle* AOC and is perpendicular to OA.

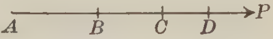
If r rotates still further, to the position OD, it has then generated the *obtuse angle* AOD.

If r rotates to the position OE, it has then generated the *straight angle* AOE.

If r rotates to the position OF, it has then generated the *reflex angle* AOF.

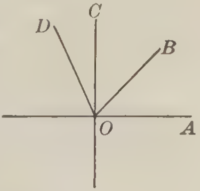
If r rotates still further, past OG to the position OA again, it has made a complete revolution and has generated two straight angles or four right angles.

40. Sums and Differences of Magnitudes. If the straight line AP has been generated by a point P moving from A to P, the segments AB, BC, CD, and so on, having been generated in succession, then we call AC the *sum* of AB and BC. That is,



$$AC = AB + BC, \text{ whence } AC - BC = AB.$$

If the angle AOD has been generated by the line OA revolving about O as a vertex from the position OA, the angles AOB, BOC, and COD having been generated in succession, then we call angle AOC the *sum* of angles AOB and BOC. That is, considering angles,



$$AOC = AOB + BOC, \text{ whence } AOC - BOC = AOB.$$

In the same way that we may have the sum or the difference of lines or of angles we may have the sum or the difference of surfaces or of solids.

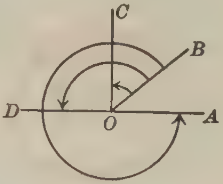
41. Perigon. The whole angular space in a plane about a point is called a *perigon.*

It therefore follows that a perigon equals the sum of two straight angles or the sum of four right angles.

42. Complements, Supplements, and Conjugates. If the sum of two angles is a right angle, each angle is called the *complement* of the other.

If the sum of two angles is a straight angle, each angle is called the *supplement* of the other.

If the sum of two angles is a perigon, each angle is called the *conjugate* of the other.



Thus, with respect to angle AOB,
the *complement* is angle BOC,
the *supplement* is angle BOD,
the *conjugate* is angle BOA (reflex).

43. Properties of Supplementary Angles. It is sufficiently evident to be taken without proof that

1. *The two adjacent angles which one straight line makes with another are together equal to a straight angle.*

2. *If the sum of two adjacent angles is a straight angle, their exterior sides are in the same straight line.*

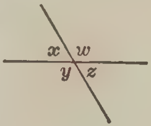
44. Angle Measure. Angles are measured by taking as a unit $\frac{1}{360}$ of a perigon. This unit is called a *degree.*

The degree is divided into 60 equal parts, called *minutes*, and the minute into 60 equal parts, called *seconds.*

We write 5° 13′ 12″ for 5 degrees 13 minutes 12 seconds.

It is evident that a right angle equals 90°, a straight angle equals 180°, and a perigon equals 360°.

45. Vertical Angles. When two angles have the same vertex, and the sides of the one are prolongations of the sides of the other, those angles are called *vertical angles.*



In the figure the angles x and z are vertical angles, as are also the angles w and y.

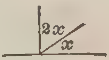
EXERCISE 3

1. Find the complement of 72°; of 65° 30′; of 22° 20′ 15″.

2. What is the supplement of 45°? of 120°? of 145° 5′? of 22° 20′ 15″?

3. What is the conjugate of 240°? of 280°? of 312° 10′ 40″?

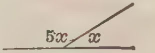
4. The complement of a certain angle x is $2x$. How many degrees are there in x?



5. The complement of a certain angle x is $3x$. How many degrees are there in x?

6. What is the angle of which the complement is four times the angle itself?

7. The supplement of a certain angle x is $5x$. How many degrees are there in x?

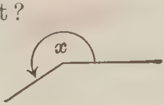


8. The supplement of a certain angle x is $14x$. How many degrees are there in x?

9. What is the angle of which the supplement equals half of the angle itself?

10. How many degrees in an angle that equals its own complement? in one that equals its own supplement?

11. The conjugate of a certain angle x is $\frac{2}{3}x$. How many degrees are there in x?



12. The conjugate of a certain angle x is $\frac{1}{2}x$. How many degrees are there in x?

13. How many degrees in an angle that equals a third of its own conjugate? in one that equals its own conjugate?

14. Find two angles, x and y, such that their sum is 90° and their difference is 10°.

15. Find two complementary angles such that their difference is 30°.

16. Find two supplementary angles such that one is 20° greater than the other.

17. The angles x and y are conjugate angles, and their difference is a straight angle. How many degrees are there in each?

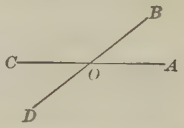
18. The angles x and y are conjugate angles, and their difference is zero. How many degrees are there in each?

19. Of two complementary angles one is four fifths of the other. How many degrees are there in each?

20. Of two supplementary angles one is five times the other How many degrees are there in each?

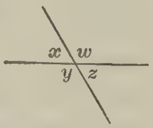
21. How many degrees are there in the smaller angle formed by the hands of a clock at 5 o'clock?

22. How many degrees are there in the smaller angle formed by the hands of a clock at 10 o'clock?



23. In this figure, if angle AOB is 38°, how many degrees in angle BOC? How many in angle COD? How many in angle DOA?

24. In the same figure, if angle AOB is equal to a third of angle BOC, how many degrees in each of the four angles?



25. In the angles of this figure, if $w = 2x$, how many degrees in each? How many degrees in y? How many degrees in z?

26. Find the angle whose complement decreased by 30° equals the angle itself.

27. Find the angle whose complement divided by 2 equals the angle itself.

28. Draw a figure to show that if two adjacent angles have their exterior sides in the same straight line, their sum is a straight angle.

29. Draw a figure to show that the sum of all the angles on the same side of a straight line, at a given point, is equal to two right angles.

30. Draw a figure to show that the complements of equal angles are equal.

46. Axiom. A general statement admitted without proof to be true is called an *axiom.*

For example, it is stated in algebra that "if equals are added to equals the sums are equal." This is so simple that it is generally accepted without proof. It is therefore an axiom.

47. Postulate. In geometry a geometric statement admitted without proof to be true is called a *postulate.*

For example, it is so evident that all straight angles are equal, that this statement is a postulate. It is also evident that a straight line may be drawn and that a circle may be described, and these statements are therefore postulates of geometry.

Axioms are therefore general mathematical assumptions, while geometric postulates are the assumptions peculiar to geometry. Postulates and axioms are the assumptions upon which the whole science of mathematics rests.

48. Theorem. A statement to be proved is called a *theorem.*

For example, it is stated in arithmetic that the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides. This statement is a theorem to be proved in geometry.

49. Problem. A construction to be made so that it shall satisfy certain given conditions is called a *problem.*

For example, required to construct a triangle all of whose sides shall be equal. This construction was made in § 31, Ex. 4, and later it will be proved that the construction was correct.

50. Proposition. A statement of a theorem to be proved or a problem to be solved is called a *proposition*

In geometry, therefore, a proposition is either a theorem or a problem. We shall find that most of the propositions at first are theorems. After we have proved a number of theorems so that we can prove that the solutions of problems are correct, we shall solve some problems.

51. Corollary. A truth that follows from another with little or no proof is called a *corollary.*

For example, since we admit that all straight angles are equal, it follows as a corollary that all right angles are equal, since a right angle is half of a straight angle.

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52. Axioms. The following are the most important axioms used in geometry:

1. *If equals are added to equals the sums are equal.*
2. *If equals are subtracted from equals the remainders are equal.*
3. *If equals are multiplied by equals the products are equal.*
4. *If equals are divided by equals the quotients are equal.*

In division the divisor is never zero.

5. *Like powers or like positive roots of equals are equal.*

We learn from algebra that the square root of 4 is $+2$ or -2, but of course these are not equal. In geometry we shall use only the positive roots.

6. *If unequals are operated on by positive equals in the same way, the results are unequal in the same order.*

Taking $a > b$ and taking x and y as equal positive quantities, this axiom states that

$$a + x > b + y, \quad a - x > b - y, \quad ax > by, \quad \frac{a}{x} > \frac{b}{y}, \text{ etc.}$$

7. *If unequals are added to unequals in the same order, the sums are unequal in the same order; if unequals are subtracted from equals the remainders are unequal in the reverse order.*

If $a > b$, $c > d$, and $x = y$, then $a + c > b + d$, and $x - a < y - b$.

8. *Quantities that are equal to the same quantity or to equal quantities are equal to each other.*

9. *A quantity may be substituted for its equal in an equation or in an inequality.*

Thus if $x = b$ and if $a + x = c$, then $a + b = c$; and if $a + x > c$, then $a + b > c$. Axiom 8 is used so often that it is stated separately, although it is really included in Axiom 9.

10. *If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.*

Thus if $a > b$, and if $b > c$, then $a > c$.

11. *The whole is greater than any of its parts, and is equal to the sum of all of its parts.*

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53. Postulates. The following are among the most important postulates used in geometry. Others will be introduced as needed.

1. *One straight line and only one can be drawn through two given points.*

2. *A straight line may be produced to any required length.*

To produce AB means to extend it through B; to produce BA means to extend it through A.

3. *A straight line is the shortest path between two points.*

4. *A circle may be described with any given point as a center and any given line as a radius.*

5. *Any figure may be moved from one place to another without altering its size or shape.*

6. *All straight angles are equal.*

54. Corollary 1. *Two points determine a straight line.*

This is only a brief way of stating Postulate 1.

55. Corollary 2. *Two straight lines can intersect in only one point.*

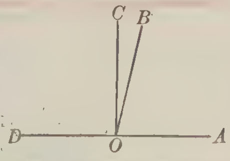
For if they had two points in common they would coincide (Post. 1).

56. Corollary 3. *All right angles are equal.*

For all straight angles are equal (Post. 6), and a straight angle (§ 34) is twice a right angle. Hence Axiom 4 applies.

57. Corollary 4. *From a given point in a given line only one perpendicular can be drawn to the line.*

For if there could be two perpendiculars to DA at O, as OB and OC, we should have angles AOB and AOC both right angles, which is impossible (§ 56).



58. Corollary 5. *Equal angles have equal complements, equal supplements, and equal conjugates.*

59. Corollary 6. *The greater of two angles has the less complement, the less supplement, and the less conjugate.*

EXERCISE 4

1. If $10° + \angle x = 27° \, 30'$, find the value of $\angle x$.

2. If $\angle x + 37° = \frac{1}{2} \angle x + 40°$, find the value of $\angle x$.

3. If $\frac{2}{3} \angle x + \angle b = 5 \angle b$, find the value of $\angle x$.

4. If $\angle x + \angle a = 4 \angle a - \angle x$, find the value of $\angle x$.

Find the value of $\angle x$ *in each of the following equations:*

5. $\angle x + 13° = 39°$.

6. $\angle x - 17° = 46°$.

7. $2 \angle x = \angle x + 23°$.

8. $5 \angle x = 2 \angle x + 21°$.

9. $4 \angle x = \frac{1}{2} \angle x + 70°$.

10. $\angle x = 0.7 \angle x + 33°$.

11. $\angle x = 0.1 \angle x + 18°$.

12. $\frac{3}{4} \angle x = \frac{1}{2} \angle x + 2\frac{1}{2}°$.

13. $\frac{4}{5} \angle x = 0.1 \angle x + 14°$.

14. $\frac{2}{3} \angle x = \frac{1}{2} \angle x + 2°$.

15. $12 \angle x + 17° = 9 \angle x + 32°$.

16. $5 \angle x - 22° \, 30' = 2 \angle x + 11°$.

17. $51° \, 20' - \frac{2}{3} \angle x = 5° \, 1' + 3 \angle x$.

18. $73° \, 21' \, 4'' - \angle x = 3° \, 3' \, 12'' + 4 \angle x$.

19. If $x + 20° = y$ and $y - 5° = 2x$, what is the value of x and of y?

Find the value of x *and of* y *in each of the following sets of equations:*

20. $x + y = 45°$,
$x - y = 35°$.

21. $x - 8y = 0°$,
$x + 8y = 80°$.

22. $2x + y = 64°$,
$3x - y = 88°$.

23. $x + 2y = 21°$,
$x + 3y = 26° \, 15'$.

24. $x + y = 9° \, 20' \, 15''$,
$2x - y = 12° \, 25' \, 15''$.

25. $x - y = 5' \, 5''$,
$3x + 4y = 14° \, 50' \, 50''$.

26. If $x < 10°$ and $y = 7° \, 30'$, what can be said as to the value of $x + y$?

27. In Ex. 26, what can be said as to the value of $x - y$?

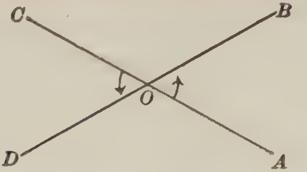
BOOK I

RECTILINEAR FIGURES

PROPOSITION I. THEOREM

定理

60. *If two lines intersect, the vertical angles are equal.*



Given the lines *AC* and *BD* intersecting at *O*.

To prove that $\angle AOB = \angle COD$.

Proof. $\angle AOB + \angle BOC =$ a st. $\angle$. § 43

(*The two adjacent angles which one straight line makes with another are together equal to a straight angle.*)

Likewise $\angle BOC + \angle COD =$ a st. $\angle$. § 43

$\therefore \angle AOB + \angle BOC = \angle BOC + \angle COD$. Post. 6

(*All straight angles are equal.*)

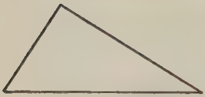
$\therefore \angle AOB = \angle COD$. Ax. 2

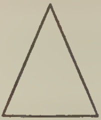
(*If equals are subtracted from equals the remainders are equal.*) Q. E. D.

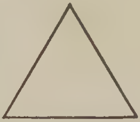
61. Nature of a Proof. From Prop. I it is seen that a theorem has (1) certain things *given*; (2) a definite thing *to be proved*; (3) a *proof*, consisting of definite statements, each supported by the authority of a definition, an axiom, a postulate, or some proposition previously proved.

62. Triangles classified as to Sides. A triangle is said to be

scalene when no two of its sides are equal;
isosceles when two of its sides are equal;
equilateral when all of its sides are equal.

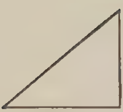

Scalene

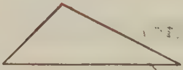

Isosceles

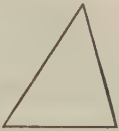

Equilateral

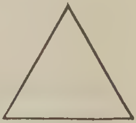
63. Triangles classified as to Angles. A triangle is said to be

right when one of its angles is a right angle;
obtuse when one of its angles is an obtuse angle;
acute when all of its angles are acute angles;
equiangular when all of its angles are equal.


Right


Obtuse


Acute


Equiangular

64. Corresponding Angles and Sides. If two triangles have the angles of the one respectively equal to the angles of the other, the equal angles are called *corresponding angles*, and the sides opposite these angles are called *corresponding sides.*

Corresponding parts are also called *homologous parts.* 相同的

65. Square. A rectilinear figure having four equal sides and four right angles is called a *square.*

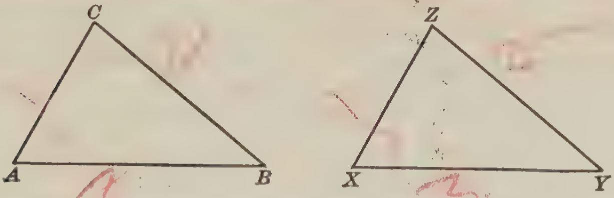
66. Congruent. If two figures can be made to coincide in all their parts, they are said to be *congruent.*

67. COROLLARY. *Corresponding parts of congruent figures are equal.*

When equal figures are necessarily congruent, as in the case of angles or straight lines, the word *equal* is used. For symbols see page vi.

PROPOSITION II. THEOREM

68. *Two triangles are congruent if two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.*



Given the triangles *ABC* and *XYZ*, with *AB* equal to *XY*, *AC* equal to *XZ*, and the angle *A* equal to the angle *X*.

To prove that $\triangle ABC$ *is congruent to* $\triangle XYZ$.

Proof. Place the $\triangle ABC$ upon the $\triangle XYZ$ so that A shall fall on X and AB shall fall along XY. Post. 5

(*Any figure may be moved from one place to another without altering its size or shape.*)

Then B will fall on Y,

(*For AB is given equal to XY.*)

AC will fall along XZ,

(*For $\angle A$ is given equal to $\angle X$.*)

and C will fall on Z.

(*For AC is given equal to XZ.*)

$\therefore$ CB will coincide with ZY. Post. 1

(*One straight line and only one can be drawn through two given points.*)

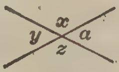
$\therefore$ the two $\triangle$ coincide and are congruent, by § 66. Q.E.D.

69. COROLLARY. *Two right triangles are congruent if the sides of the right angles are equal respectively.*

The right angles are equal (§ 56). How does Prop. II apply?

EXERCISE 5

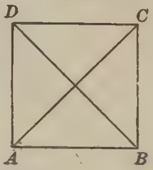
1. In this figure if $\angle a = 53°$, how many degrees are there in $\angle y$? in $\angle x$? in $\angle z$?



2. In Ex. 1, if $\angle a$ were increased to 89°, what would then be the size of $\angle$s x, y, and z?

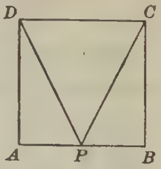
3. In the square $ABCD$, prove that $AC = BD$.

In $\triangle$s ABC and BAD what two sides of the one are known to be equal to what two sides of the other? How about the included angles? Write a complete proof as in Prop. II.



4. If $ABCD$ is a square and P is the mid-point of AB, prove that $PC = PD$.

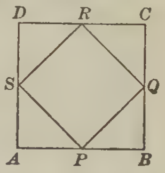
What triangles should be proved congruent? Can this be done by Prop. II? Write the proof.



5. How many degrees in an angle that equals one fourth of its complement? one tenth of its complement?

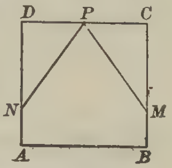
6. How many degrees in an angle that equals twice its supplement? one third of its supplement?

7. In the square $ABCD$ the points P, Q, R, S bisect the consecutive sides. Prove that $PQ = QR = RS = SP$.

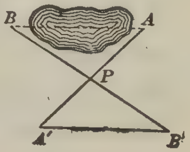


8. In the square $ABCD$ the point P bisects CD, and BM is made equal to AN, as shown in this figure. Prove that $PM = PN$.

What two sides and included angle of one triangle must be proved equal to what two sides and included angle of another triangle?



9. Prove that to determine the distance AB across a pond one may sight from A across a post P, place a stake at A' making $PA' = AP$, then sight along BP making $PB' = BP$, and finally measure $A'B'$.



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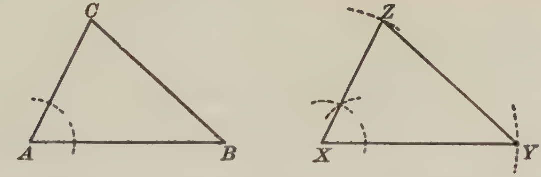
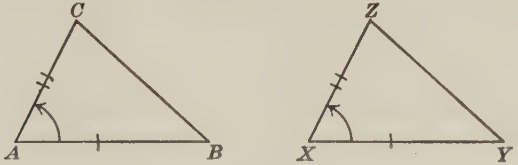
70. Drawing the Figures. Directions have already been given (§ 31) for drawing the most common geometric figures. For example, in Prop. II the complete work of drawing $\triangle XYZ$ so that $XY = AB$, $\angle X = \angle A$, and $XZ = AC$, is indicated in the following figures, *the construction lines being dotted*, as is always the case in this book.



It is desirable to construct such figures accurately, employing compasses and ruler until such time as the use of these instruments is thoroughly understood. Eventually, however, the figures should be rapidly but neatly drawn, free-hand or with the aid of the ruler, as the mathematician usually makes his figures.

71. Designating Corresponding Sides and Angles. It is helpful in propositions concerning equality of figures to check the equal parts so that the eye can follow the proof more easily. Thus it would be convenient to represent the above figures as follows:



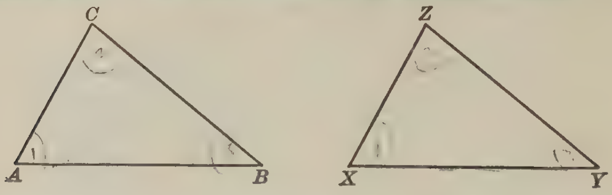
Here AB and XY have one check, AC and XZ two checks, and the equal angles A and X are marked by curved arrows.

If a figure is very complicated, there is sometimes an advantage in using colored crayons or colored pencils, but otherwise this expedient is of little value.

While such figures have some attraction for the eye they are not generally used in practice, one reason being that the student rarely has a supply of colored pencils at hand when studying by himself.

定理→PROPOSITION III. THEOREM

72. *Two triangles are congruent if two angles and the included side of the one are equal respectively to two angles and the included side of the other.*



Given the triangles *ABC* and *XYZ*, with angle *A* equal to angle *X*, angle *B* equal to angle *Y*, and with *AB* equal to *XY*.

To prove that $\triangle ABC$ *is congruent to* $\triangle XYZ$.

Proof. Place the $\triangle ABC$ upon the $\triangle XYZ$ so that AB shall coincide with its equal, XY. Post. 5

(*Any figure may be moved from one place to another without altering its size or shape.*)

Then AC will fall along XZ and BC along YZ.

(*For it is given that* $\angle A = \angle X$ *and* $\angle B = \angle Y$.)

$\therefore$ C will fall on Z. § 55

(*Two straight lines can intersect in only one point.*)

$\therefore$ the two $\triangle$ are congruent. § 66

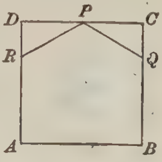
(*If two figures can be made to coincide in all their parts, they are said to be congruent.*) Q. E. D.

73. Hypothesis. A supposition made in an argument is called an *hypothesis*.

Thus, where it is said that $\angle A = \angle X$ and $\angle B = \angle Y$, we might say that this is true "by hypothesis," instead of saying that $\angle A$ is given equal to $\angle X$, and $\angle B$ is given equal to $\angle Y$. The word is generally used, however, for an assumption made somewhere *in the proof*.

EXERCISE 6

1. In the square $ABCD$ the point P bisects CD, and PQ and PR are drawn so that $\angle QPC = 30°$ and $\angle RPQ = 120°$. Prove that $PQ = PR$.

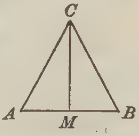


If $\angle QPC = 30°$ and $\angle RPQ = 120°$, what does $\angle DPR$ equal?

In the two triangles what parts are respectively equal, and why?

Write the proof in full.

2. In this figure prove that if CM bisects $\angle ACB$ and is also perpendicular to AB, the triangle ABC is isosceles.



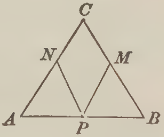
In $\triangle AMC$ and BMC are two angles of the one respectively equal to two angles of the other? Why?

The two triangles have one common side.

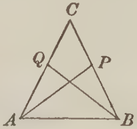
Write the proof in full.

3. In the triangle ABC, $AC = BC$ and CM bisects the angle C. Prove that CM bisects the base AB.

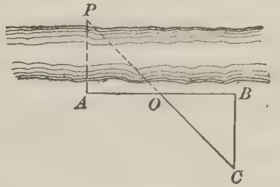
4. The triangle ABC has $\angle A$ equal to $\angle B$. The point P bisects AB, and the lines PM and PN are drawn so that $\angle BPM = \angle NPA$. Prove that $BM = AN$.



5. The triangle ABC has $\angle A = \angle B$. The lines AP and BQ are so drawn that $\angle BAP = \angle QBA$. Prove that $AP = BQ$.



6. Wishing to measure the distance across a river, some boys sighted from A to a point P. They then turned and measured AB at right angles to AP. They placed a stake at O, halfway from A to B, and drew a perpendicular to AB at B. They placed a stake at C, on this perpendicular, and in line with O and P. They then found the width by measuring BC. Prove that they were right.



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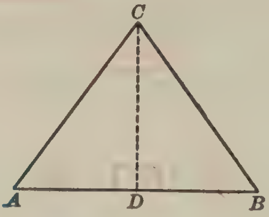
PROPOSITION IV. THEOREM

74. *In an isosceles triangle the angles opposite the equal sides are equal.*

二等边
三角形



Given the isosceles triangle ABC, with AC equal to BC.

To prove that $\angle A = \angle B$.

Proof. Suppose CD drawn so as to bisect $\angle ACB$.

Then in the $\triangle$ ADC and BDC,

$AC = BC$, Given

$CD = CD$, Iden.

(*That is, CD is common to the two triangles.*)

and $\angle ACD = \angle DCB$. Hyp.

(*For CD bisects $\angle ACB$.*)

$\therefore \triangle ADC$ is congruent to $\triangle BDC$. § 68

(*Two $\triangle$ are congruent if two sides and the included $\angle$ of the one are equal respectively to two sides and the included $\angle$ of the other.*)

$\therefore \angle A = \angle B$. § 67

(*Corresponding parts of congruent figures are equal.*) Q.E.D.

This proposition has long been known as the *Pons asinorum*, or Bridge of Fools (asses). It is attributed to Thales, a Greek philosopher.

In an isosceles triangle the side which is not one of the two equal sides is called the *base*.

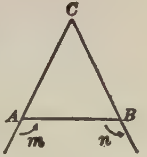
75. COROLLARY. *An equilateral triangle is equiangular.*

Is an equilateral triangle a special kind of isosceles triangle?

EXERCISE 7

1. With the figure of Prop. IV, if $AC = BC$ and CD bisects $\angle C$, prove that CD is $\perp$ to AB.

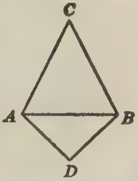
What angles must be proved to be right angles? What is a right angle? Do these angles fulfill the requirements of the definition?



2. In the adjacent figure $AC = BC$. Prove that $\angle m = \angle n$.

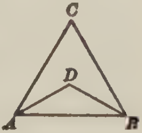
3. In the following figure $AC = BC$ and $AD = BD$. Prove that $\angle CBD = \angle DAC$.

What angles are equal by Prop. IV? Then what axiom applies?



4. In the figure of Ex. 3 prove that if a line is drawn from C to D, the $\triangle DBC$ is congruent to the $\triangle DAC$.

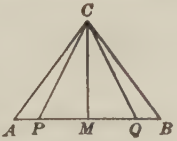
5. Two isosceles triangles, ABC and ABD, are constructed on the same side of the common base AB. Prove that $\angle CBD = \angle DAC$.



6. In the figure of Ex. 5 prove that a line drawn through C and D bisects $\angle ADB$.

What two triangles must be proved congruent?

7. In this figure $AC = BC$ and $AP = BQ$. Prove that $PC = QC$. Also prove that $\angle MPC = \angle CQM$.

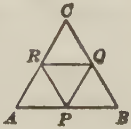


8. In this figure, if $AC = BC$, $AP = BQ$, and $PM = QM$, prove that CM is $\perp$ to PQ.

What angles must be proved to be right angles?

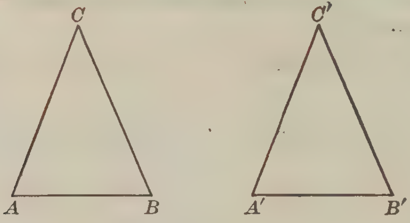
9. In this figure P, Q, and R are mid-points of the sides of the equilateral triangle ABC. Prove that PQR is an equilateral triangle.

Prove that $\triangle$s APR, BQP, and CRQ are congruent by using two propositions already proved.



PROPOSITION V. THEOREM

76. *If two angles of a triangle are equal, the sides opposite the equal angles are equal, and the triangle is isosceles.*



Given the triangle ABC, with the angle A equal to the angle B.

To prove that $AC = BC$.

Proof. Suppose the second triangle $A'B'C'$ to be an exact reproduction of the given triangle ABC.

Turn the triangle $A'B'C'$ over and place it upon ABC so that B' shall fall on A and A' shall fall on B. Post. 5

Then $B'A'$ will coincide with AB. Post. 1

Since $\angle A' = \angle B'$, Given

and $\angle A = \angle A'$, Hyp.

$\therefore \angle A = \angle B'$. Ax. 8

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$\therefore B'C'$ will lie along AC.

Similarly $A'C'$ will lie along BC.

Therefore C' will fall on both AC and BC, and hence at their intersection.

$\therefore B'C' = AC$.

But $B'C'$ was made equal to BC.

$\therefore AC = BC$, by Ax. 8. Q.E.D.

77. COROLLARY. *An equiangular triangle is equilateral.*

78. Kinds of Proof. In the five propositions thus far proved in the text two different kinds of proof have been seen:

(1) *Synthetic.* In Prop. I we put together some known truths in order to obtain a new truth. Such a method of proof is known as the *synthetic method*, and is the most common of all that are used in geometry.

In this method we endeavor simply to find what propositions have already been proved that will lead to the proof of the proposition that is before us. This method was used in all the exercises on pages 28, 31, and 33.

(2) *By superposition.* In Props. II and III we placed one figure on another and then, by synthetic reasoning, showed them to be identically equal. Such proof is known as a proof *by superposition.* Superposition means "placing on," and one figure is said to be superposed on the other.

In Prop. V a special kind of proof by superposition was employed, in which we superpose a figure on its exact duplicate. This special method is rarely used, but in this proposition it materially simplifies the proof.

79. Converse Propositions. If two propositions are so related that what is given in each is what is to be proved in the other, each proposition is called the *converse* of the other.

E.g. in Prop. IV we have given

$$AC = BC, \text{ to prove that } \angle A = \angle B.$$

In Prop. V we have given

$$\angle A = \angle B, \text{ to prove that } AC = BC.$$

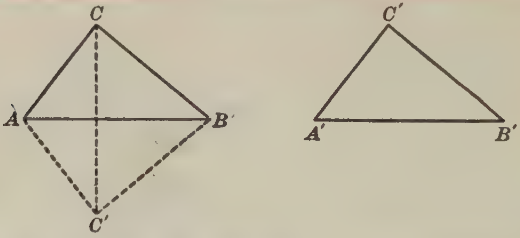
Hence Prop. V is the converse of Prop. IV, and Prop. IV is the converse of Prop. V.

Not all converses are true, and hence we have to prove any given converse.

E.g. the converse of the statement "Two right angles are two equal angles" is "Two equal angles are two right angles," and this statement is evidently false.

PROPOSITION VI. THEOREM

80. *Two triangles are congruent if the three sides of the one are equal respectively to the three sides of the other.*



Given the triangles *ABC* and *A'B'C'*, with *AB* equal to *A'B'*, *AC* equal to *A'C'*, and *BC* equal to *B'C'*.

To prove that $\triangle ABC$ *is congruent to* $\triangle A'B'C'$.

Proof. Let AB and $A'B'$ be the greatest of the sides of the $\triangle$s.

Place $\triangle A'B'C'$ next to $\triangle ABC$ so that A' shall fall on A, the side $A'B'$ shall fall along AB, and the vertex C' shall be opposite the vertex C. Post. 5

Then B' will fall on B.

(For $A'B'$ is given equal to AB.)

Draw CC'.

Since $AC = AC'$, Given

$\therefore \angle ACC' = \angle CC'A$. § 74

Since $BC = BC'$, Given

$\therefore \angle C'CB = \angle BC'C$. § 74

$\therefore \angle ACC' + \angle C'CB = \angle CC'A + \angle BC'C$. Ax. 1

Hence $\angle ACB = \angle BC'A$. Ax. 11

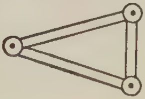
(For $\angle ACB$ is made up of $\angle ACC'$ and $\angle C'CB$, and $\angle BC'A$ is made up of $\angle CC'A$ and $\angle BC'C$.)

$\therefore \triangle ABC$ is congruent to $\triangle ABC'$. § 68

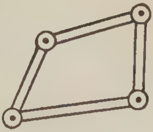
$\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$, by Ax. 9. Q.E.D.

EXERCISE 8

1. Prove that a line from the vertex to the mid-point of the base of an isosceles triangle cuts the triangle into two congruent triangles.

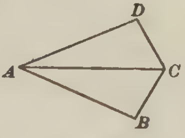


2. Three iron rods are hinged at the extremities, as shown in this figure. Is the figure rigid? Why?

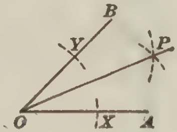


3. Four iron rods are hinged, as shown in this figure. Is the figure rigid? If not, where would you put in the fifth rod to make it rigid? Prove that this would accomplish the result.

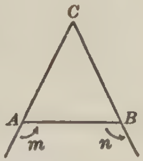
4. If two isosceles triangles are constructed on opposite sides of the same base, prove by Prop. VI and § 67 that the line through the vertices bisects the vertical angles.



5. In this figure $AB = AD$ and $CB = CD$. Prove that AC bisects $\angle BAD$ and $\angle DCB$.



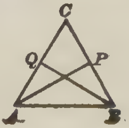
6. In § 31, Ex. 8, it was shown how to bisect an angle, this being the figure used. Draw PX and PY, and prove by Prop. VI that PO bisects $\angle AOB$.



7. In a triangle ABC it is known that $AC = BC$. If $\angle A$ and $\angle B$ are both bisected by lines meeting at P, prove that $\triangle ABP$ is isosceles.

8. In this figure it is known that $\angle m = \angle n$. Prove that $AC = BC$.

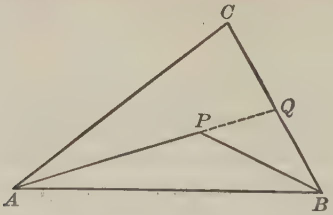
9. From the vertices A and B of an equilateral triangle lines are drawn to the mid-points of the opposite sides. Prove that these two lines are equal.



In $\triangle$ ABQ and BAP show that the conditions of congruence as stated in Prop. II are fulfilled.

PROPOSITION VII. THEOREM

81. *The sum of two lines from a given point to the extremities of a given line is greater than the sum of two other lines similarly drawn, but included by them.*



Given CA and CB, two lines drawn from the point C to the extremities of the line AB, and PA and PB two lines similarly drawn, but included by CA and CB.

To prove that $CA + CB > PA + PB$.

Proof. Produce AP to meet the line CB at Q. Post. 2

Then $CA + CQ > PA + PQ$. Post. 3

(*A straight line is the shortest path between two points.*)

Likewise $BQ + PQ > PB$. Post. 3

Add these inequalities, and we have

$$CA + CQ + BQ + PQ > PA + PQ + PB. \qquad \text{Ax. 7}$$

(*If unequals are added to unequals in the same order, the sums are unequal in the same order.*)

Substituting for $CQ + BQ$ its equal CB, we have

$$CA + CB + PQ > PA + PQ + PB. \qquad \text{Ax. 9}$$

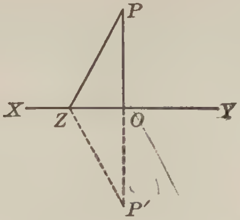
(*A quantity may be substituted for its equal in an equation or in an inequality.*)

Taking PQ from each side of the inequality, we have

$CA + CB > PA + PB$, by Ax. 6. Q. E. D.

PROPOSITION VIII. THEOREM

82. *Only one perpendicular can be drawn to a given line from a given external point.*



Given a line XY, P an external point, PO a perpendicular to XY from P, and PZ any other line from P to XY.

To prove that PZ *is not* $\perp$ *to* XY.

Proof. Produce PO to P', making OP' equal to PO. Post. 2

Draw $P'Z$. Post. 1

By construction POP' is a straight line.

$\therefore PZP'$ is not a straight line. Post. 1

Hence $\angle P'ZP$ is not a straight angle. § 33

Since $\angle$s POZ and ZOP' are rt. $\angle$s, § 27

$\therefore \angle POZ = \angle ZOP'$ § 56

Furthermore $PO = OP'$, Hyp.

and $OZ = OZ$. Iden.

$\therefore \triangle OPZ$ is congruent to $\triangle OP'Z$, § 68

(*Two $\triangle$s are congruent if two sides and the included $\angle$ of the one are equal respectively to two sides and the included $\angle$ of the other.*)

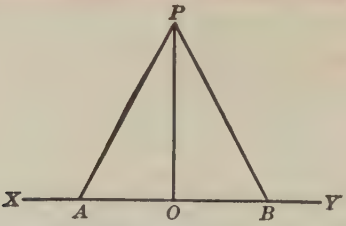
and $\angle OZP = \angle P'ZO$. § 67

$\therefore \angle OZP$, the half of $\angle P'ZP$, is not a right angle. § 34

$\therefore PZ$ is not $\perp$ to XY, by § 27. Q.E.D.

PROPOSITION IX. THEOREM

83. *Two lines drawn from a point in a perpendicular to a given line, cutting off on the given line equal segments from the foot of the perpendicular, are equal and make equal angles with the perpendicular.*



Given PO perpendicular to XY, and PA and PB two lines cutting off on XY equal segments OA and OB from O.

To prove that $PA = PB$,

and $\angle APO = \angle OPB$.

Proof. In the $\triangle$s AOP and BOP,

$\angle POA$ and $\angle BOP$ are rt. $\angle$s. § 27

(*For PO is given $\perp$ to XY.*)

$\therefore \angle POA = \angle BOP$. § 56

(*All right $\angle$s are equal.*)

Also $OA = OB$, Given

and $PO = PO$. Iden.

(*That is, PO is common to the two $\triangle$s.*)

$\therefore \triangle AOP$ is congruent to $\triangle BOP$. § 68

(*Two $\triangle$s are congruent if two sides and the included $\angle$ of the one are equal respectively to two sides and the included $\angle$ of the other.*)

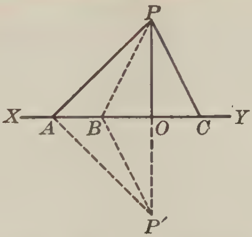
$\therefore PA = PB$,

and $\angle APO = \angle OPB$. § 67

(*Corresponding parts of congruent figures are equal.*) Q. E. D.

PROPOSITION X. THEOREM

84. *Of two lines drawn from a point in a perpendicular to a given line, cutting off on the given line unequal segments from the foot of the perpendicular, the more remote is the greater.*



Given PO perpendicular to XY, PA and PC two lines drawn from P to XY, and OA greater than OC.

To prove that $PA > PC$.

Proof. Take OB equal to OC, and draw PB.

Then $PB = PC$. § 83

Produce PO to P', making $OP' = PO$, and draw $P'A$ and $P'B$.

Then $PA = P'A$ and $PB = P'B$. § 83

But $PA + P'A > PB + P'B$. § 81

$\therefore 2\,PA > 2\,PB$ and $PA > PB$. Axs. 9 and 6

$\therefore PA > PC$, by Ax. 9. Q.E.D.

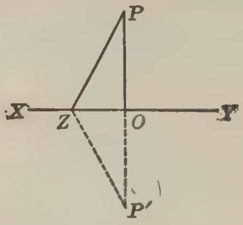
85. COROLLARY. *Only two equal obliques can be drawn from a given point to a given line, and these cut off equal segments from the foot of the perpendicular.*

Of two unequal lines from a point to a line, the greater cuts off the greater segment from the foot of the perpendicular.

For $PB = PC$, but PB cannot equal PA (§ 84). The segments OB and OC are equal, for otherwise PB could not equal PC.

PROPOSITION XI. THEOREM

86. *The perpendicular is the shortest line that can be drawn to a given line from a given external point.*



Given a line XY, P an external point, PO the perpendicular, and PZ any other line drawn from P to XY.

To prove that $PO < PZ$.

Proof. Produce PO to P', making $OP' = PO$; and draw $P'Z$.

Then $PZ = P'Z$. § 83

(*Two lines drawn from a point in a* ⊥ *to a given line, cutting off on the given line equal segments from the foot of the* ⊥, *are equal.*)

$\therefore PZ + P'Z = 2\,PZ$. Ax. 1

Furthermore $PO + P'O = 2\,PO$. Ax. 1

But $PO + P'O < PZ + P'Z$. Post. 3

$\therefore 2\,PO < 2\,PZ$. Ax. 9

$\therefore PO < PZ$, by Ax. 6. Q.E.D.

87. Hypotenuse. The side opposite the right angle in a right triangle is called the *hypotenuse.*

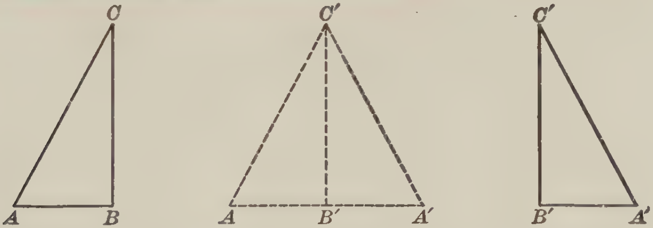
The other two sides of a right triangle are usually called the *sides.*

88. Distance. The length of the straight line from one point to another is called the *distance* between the points.

The length of the perpendicular from an external point to a line is called the *distance* from the point to the line.

Proposition XII. Theorem

89. *Two right triangles are congruent if the hypotenuse and a side of the one are equal respectively to the hypotenuse and a side of the other.*



Given the right triangles ABC and $A'B'C'$, with the hypotenuse AC equal to the hypotenuse $A'C'$, and with BC equal to $B'C'$.

To prove that $\triangle ABC$ *is congruent to* $\triangle A'B'C'$.

Proof. Place $\triangle ABC$ next to $\triangle A'B'C'$, so that BC shall fall along $B'C'$, B shall fall on B', and A and A' shall fall on opposite sides of $B'C'$. Post. 5

Then C will fall on C',

(*For BC is given equal to $B'C'$.*)

and BA will fall along $A'B'$ produced. § 34

(*For $\angle CBA + \angle A'B'C' =$ a st. $\angle$.*)

Since $AC' = A'C'$,

$\therefore AB' = A'B'$. § 85

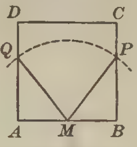
$\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$. § 80

(*Two $\triangle$s are congruent if the three sides of the one are equal respectively to the three sides of the other.*) Q.E.D.

90. Corollary. *Two right triangles are congruent if any two sides of the one are equal respectively to the corresponding two sides of the other.*

EXERCISE 9

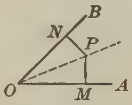
1. $ABCD$ is a square and M is the mid-point of AB. With M as a center an arc is drawn, cutting BC at P and AD at Q. Prove that $\triangle MBP$ is congruent to $\triangle MAQ$, and write the general statement of this theorem without using letters as is done here.



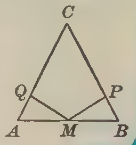
This would read, "If an arc is drawn, with the mid-point of one side of a square as a center, cutting the sides perpendicular to that side, then the triangles cut off by," etc.

2. Draw a figure similar to that of Ex. 1, but take a radius such that the arc cuts BC produced at a point above C, and AD above D. Then prove that $\triangle MBP$ is congruent to $\triangle MAQ$.

3. Prove that if from the point P the perpendiculars PM, PN to the sides of an angle AOB are equal, the point P lies on the bisector of the angle AOB. Write the general statement of this theorem without using letters as is done here.



4. Prove that if the perpendiculars from the mid-point M of the base AB of a triangle ABC to the sides of the triangle are equal, then $\angle A = \angle B$. What then follows as to the sides AC and BC? Write the general statement of this theorem without referring to a special figure.



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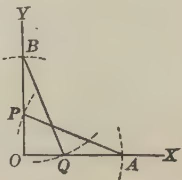
5. Prove that if the perpendiculars from the extremities of the base of a triangle to the other two sides are equal, the triangle is isosceles.

6. Suppose $OY \perp OX$. With O as a center an arc is drawn cutting OX at A and OY at B. Then with A as a center an arc is drawn cutting OY at P, and with B as a center and the same radius an arc is drawn cutting OX at Q. Prove that $OP = OQ$.

What triangles are congruent by Prop. XII?



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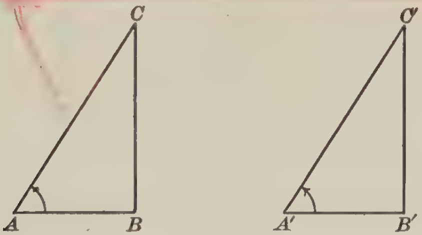
PROPOSITION XIII. THEOREM

91. *Two right triangles are congruent if the hypotenuse and an adjacent angle of the one are equal respectively to the hypotenuse and an adjacent angle of the other.*



Given the right triangles ABC, $A'B'C'$, with the hypotenuse AC equal to the hypotenuse $A'C'$, and with angle A equal to angle A'.

To prove that $\triangle ABC$ *is congruent to* $\triangle A'B'C'$.

Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that A shall fall upon A' and AC shall fall along $A'C'$. Post. 5

Then C will fall on C',

(*For AC is given equal to $A'C'$.*)

and AB will lie along $A'B'$.

(*For $\angle A$ is given equal to $\angle A'$.*)

Then because C falls on C',

and $\angle$s B and B' are rt. $\angle$s, Given

(*Since the* $\triangle$s *are given as rt.* $\triangle$s.)

$\therefore CB$ will coincide with $C'B'$. § 82

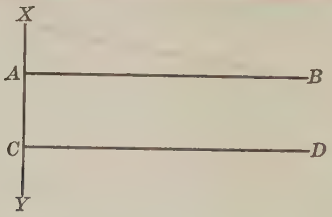
Only one perpendicular can be drawn to a given line from a given external point.)

$\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$. § 66

(*If two figures can be made to coincide in all their parts, they are said to be congruent.*) Q. E. D

Proposition XIV. Theorem

92. *Two lines in the same plane perpendicular to the same line cannot meet however far they are produced.*



Given the lines *AB* and *CD* perpendicular to *XY* at *A* and *C* respectively.

To prove that AB and CD cannot meet however far they are produced.

Proof. If AB and CD can meet if sufficiently produced, we shall have two perpendicular lines from their point of meeting to the same line.

But this is impossible. § 82

$\therefore AB$ and CD cannot meet. Q.E.D.

93. Parallel Lines. Lines that lie in the same plane and cannot meet however far produced are called *parallel lines.*

94. Postulate of Parallels. *Through a given point only one line can be drawn parallel to a given line.*

As always in such cases the word *line* means straight line.

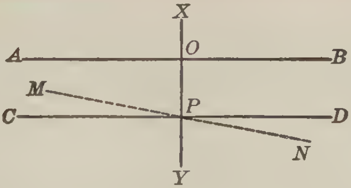
95. Corollary 1. *Two lines in the same plane perpendicular to the same line are parallel.*

96. Corollary 2. *Two lines in the same plane parallel to a third line are parallel to each other.*

For if they could meet, we should have two lines through a point parallel to a line. Why is this impossible?

PROPOSITION XV. THEOREM

97. *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*



Given AB and CD, two parallel lines, with XY perpendicular to AB and cutting CD at P.

To prove that XY *is* $\perp$ *to* CD.

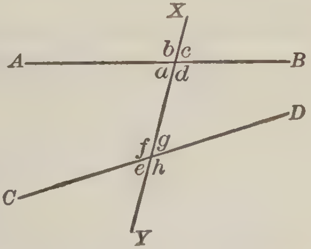
Proof. Suppose MN drawn through P $\perp$ to XY.

Then	MN is $\parallel$ to AB.	§ 95
But	CD is $\parallel$ to AB.	Given
	$\therefore$ CD and MN must coincide.	§ 94
But	XY is $\perp$ to MN.	Hyp.
	$\therefore$ XY is $\perp$ to CD.	Q. E. D.

98. Transversal. A line that cuts two or more lines is called a *transversal* of those lines.

99. Angles made by a Transversal. If XY cuts AB and CD, the angles a, d, g, f are called *interior* angles; b, c, h, e are called *exterior* angles.

The angles d and f, and a and g, are called *alternate-interior* angles; the angles b and h, and c and e, are called *alternate-exterior* angles.



The angles b and f, c and g, e and a, h and d, are called *exterior-interior* angles.

PROPOSITION XVI. THEOREM

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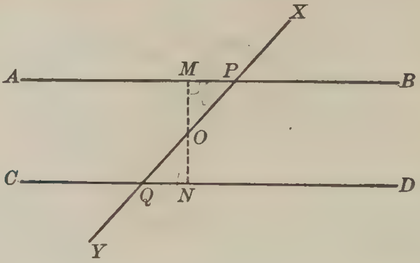
100. *If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*



Given AB and CD, two parallel lines cut by the transversal XY in the points P and Q respectively.

To prove that $\angle APQ = \angle DQP$.

Proof. Through O, the mid-point of PQ, suppose MN drawn $\perp$ to CD.

Then MN is likewise $\perp$ to AB. § 97

(*A line $\perp$ to one of two* $\parallel$*s is* $\perp$ *to the other.*)

Now $\triangle$s PMO and QNO are rt. $\triangle$s. § 63

(*Since* $\angle$s OMP *and* ONQ *are rt.* $\angle$s.)

But $\angle POM = \angle QON$, § 60

(*If two lines intersect, the vertical* $\angle$s *are equal.*)

and $OP = OQ$. Hyp.

(*For O was taken as the mid-point of PQ.*)

$\therefore \triangle PMO$ is congruent to $\triangle QNO$. § 91

(*Two right* $\triangle$s *are congruent if the hypotenuse and an adjacent* $\angle$ *of the one are equal respectively to the hypotenuse and an adjacent* $\angle$ *of the other.*)

$\therefore \angle APQ = \angle DQP$. § 67

(*Corresponding parts of congruent figures are equal.*) Q.E.D.

PROPOSITION XVII. THEOREM

101. *When two lines in the same plane are cut by a transversal, if the alternate-interior angles are equal, the two lines are parallel.*

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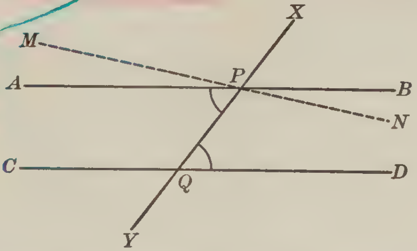

Given the lines *AB* and *CD* cut by the transversal *XY* in the points *P* and *Q* respectively, so as to make the angles *APQ* and *DQP* equal.

To prove that *AB is ∥ to CD.*

Proof. Since we do not know that AB is ∥ to CD, let us suppose MN drawn through P ∥ to CD.

We shall then prove that AB coincides with MN.

Now $\angle MPQ = \angle DQP$. § 100

(*If two ∥ lines are cut by a transversal, the alt.-int. ∠s are equal.*)

But $\angle APQ = \angle DQP$. Given

$\therefore \angle APQ = \angle MPQ$. Ax. 8

(*Quantities that are equal to the same quantity are equal to each other.*)

$\therefore AB$ and MN must coincide. § 23

(*Def. of equal angles.*)

But MN is ∥ to CD. Hyp.

(*For MN was drawn ∥ to CD.*)

$\therefore AB$, which coincides with MN, is ∥ to CD. Q.E.D.

This proposition is the converse of Prop. XVI, as defined in § 79.

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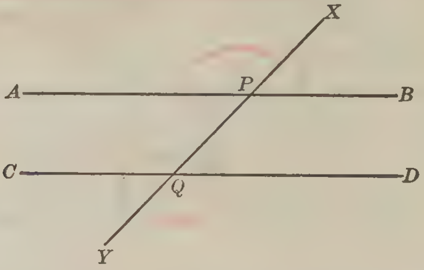
PROPOSITION XVIII. THEOREM

102. *If two parallel lines are cut by a transversal, the exterior-interior angles are equal.*



Given AB **and** CD**, two parallel lines, cut by the transversal** XY **in the points** P **and** Q **respectively.**

To prove that $\angle BPX = \angle DQX$.

Proof. $\angle BPX = \angle APQ$. § 60

$\angle APQ = \angle DQX$. § 100

$\therefore \angle BPX = \angle DQX$, by Ax. 8. Q.E.D.

103. COROLLARY 1. *When two lines are cut by a transversal, if the exterior-interior angles are equal, the lines are parallel.*

The proofs of §§ 103 and 105 are similar to that of § 101.

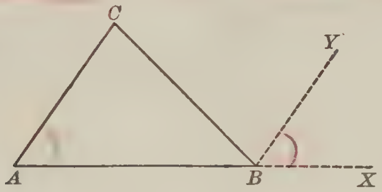
104. COROLLARY 2. *If two parallel lines are cut by a transversal, the two interior angles on the same side of the transversal are supplementary.*

105. COROLLARY 3. *When two lines are cut by a transversal, if two interior angles on the same side of the transversal are supplementary, the lines are parallel.*

106. COROLLARY 4. *If two parallel lines are cut by a transversal, the alternate-exterior angles are equal.*

PROPOSITION XIX. THEOREM

107. *The sum of the three angles of a triangle is equal to two right angles.*



Given the triangle *ABC*.

To prove that $\angle A + \angle B + \angle C = 2$ *rt.* $\measuredangle$s.

Proof. Suppose BY drawn ∥ to AC, and produce AB to X.

Then $\angle XBY + \angle YBC + \angle CBA = 2$ rt. $\measuredangle$s. § 34

(For a st. ∠ equals 2 rt. ∡s.)

But $\angle A = \angle XBY,$ § 102

and $\angle C = \angle YBC.$ § 100

$\therefore \angle A + \angle B + \angle C = 2$ rt. $\measuredangle$s, by Ax. 9. Q. E. D.

108. COROLLARY 1. *If two triangles have two angles of the one equal to two angles of the other, the third angles are equal.*

109. COROLLARY 2. *In a triangle there can be but one right angle or one obtuse angle.*

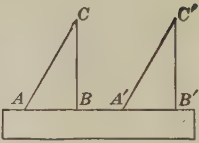
110. Exterior Angle. The angle included by one side of a figure and an adjacent side produced is called an *exterior angle.*

In the above figure $\angle XBC$ is an exterior angle, and ∡s A and C are called the *opposite interior angles.*

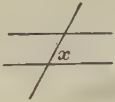
111. COROLLARY 3. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles, and is therefore greater than either of them.*

EXERCISE 10

1. Show that if we place a draftsman's triangle against a ruler and draw AC, and move the triangle along as shown in the figure and draw $A'C'$, then AC is ∥ to $A'C'$.



2. In the next figure $x = 60°$. How many degrees in each of the other seven angles?



3. In the next figure representing two pairs of parallel lines certain angles are equal. State these equalities in this form: $a = c = g = e = o = \cdots$, and give the reason in each case.

4. In the figure of Ex. 3 state ten pairs of nonadjacent angles that are supplementary. ← 附加 補
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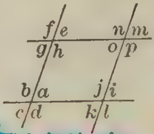
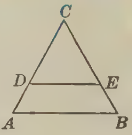
Thus: $a + h = 180°$ and $d + e = 180°$.



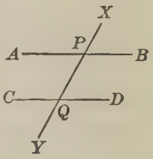
5. In the triangle ABC, $AC = BC$ and DE is drawn parallel to AB. Prove that $CD = CE$. Write a general statement of the theorem.



6. In the next figure AB is parallel to CD, and $\angle APQ$ is half of $\angle QPB$. How many degrees in the various angles?

7. If $\angle YQD = 135°$, how many degrees in the various angles?

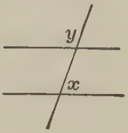
8. Let $\angle DQP = x$ and $\angle YQD = y$. Then if $y - x = 100°$, find the value of x and y.



9. Let $\angle CQY = x$ and $\angle XPA = y$. Then if $x = \frac{1}{5}y$, find the value of x and y.

10. In the next figure $x = 72°$ and $x = \frac{2}{3}y$. It is required to know if the lines are parallel, and why.

11. In the figure of Ex. 10 suppose $x = 73°$ and $y - x = 32°$. It is required to know if the lines are parallel, and why.



The three angles of a triangle are x, y, and z. Find the value of z, given the values of x and y as follows:

12. $x = 10°$, $y = 30°$.
13. $x = 20°$, $y = 20°$.
14. $x = 75°$, $y = 50°$.
15. $x = 38°$, $y = 76°$.
16. $x = 49°$, $y = 92°$.
17. $x = 37°$, $y = 48°$.
18. $x = 63°$, $y = 29°$.
19. $x = 75° 29'$, $y = 68° 41'$.
20. $x = 82° 33'$, $y = 75° 48'$.
21. $x = 69° 58'$, $y = 82° 49'$.

22. In a certain right triangle one angle is 37°. What is the size of the other acute angle?

23. In a certain right triangle one angle is 36° 41′. What is the size of the other acute angle?

24. In a certain right triangle one angle is 29° 48′ 56″. What is the size of the other acute angle?

25. In a certain right triangle one acute angle is two thirds of the other. How many degrees are there in each?

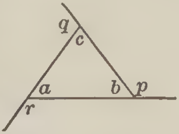
26. In a certain right triangle one acute angle is twice as large as the other. How many degrees are there in each?

27. In a certain right triangle the acute angles are $2x$ and $5x$. Find the value of x and the size of each angle.

28. In a certain triangle one angle is twice as large as another and three times as large as the third. How many degrees are there in each?

29. In a certain isosceles triangle one angle is twice another angle. How many degrees in each of the three angles?

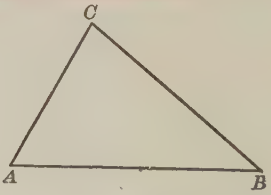
30. In this figure what single angle equals $a + c$? To the sum of what angles is q equal? also r? From these relations find the number of degrees in $p + q + r$.



31. Prove Prop. XIX by first drawing a parallel to AB through C, instead of drawing BY.

PROPOSITION XX. THEOREM

112. *The sum of any two sides of a triangle is greater than the third side, and the difference between any two sides is less than the third side.*



Given the triangle ABC, with AB the greatest side.

To prove that $BC + CA > AB$, *and* $AB - BC < CA$.

Proof. $BC + CA > AB$. Post. 3

(*A straight line is the shortest path between two points.*)

Since $BC + CA > AB$,

$\therefore CA > AB - BC$; Ax. 6

or, $AB - BC < CA$. Q.E.D.

EXERCISE 11

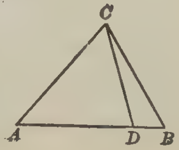
State in what cases it is possible to form triangles with rods of the following lengths, and give the reason:

1. 2 in., 3 in., 4 in.
2. 3 in., 4 in., 7 in.
3. 6 in., 7 in., 9 in.
4. 7 in., 10 in., 20 in.
5. 8 in., $9\frac{1}{2}$ in., 18 in.
6. $9\frac{3}{4}$ in., $10\frac{1}{2}$ in., $12\frac{1}{4}$ in.

7. In this figure prove that $AB + BC > AD + DC$.

Why is $DB + BC > DC$?

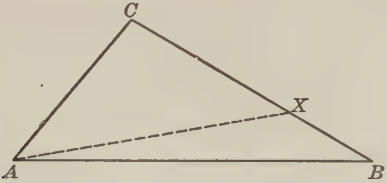
What is the result of adding AD to these unequals?



8. How many degrees are there in each angle of an equiangular triangle? Prove it.

PROPOSITION XXI. THEOREM

113. *If two sides of a triangle are unequal, the angles opposite these sides are unequal, and the angle opposite the greater side is the greater.*



Given the triangle *ABC*, with *BC* greater than *CA*.

To prove that $\angle BAC > \angle B$.

Proof. On CB suppose CX taken equal to CA.

Draw AX. Post. 1

Then $\triangle AXC$ is isosceles. § 62

Then $\angle CXA = \angle XAC$. § 74

(*In an isosceles* $\triangle$ *the* $\angle$s *opposite the equal sides are equal.*)

But $\angle CXA > \angle B$. § 111

(*An exterior* $\angle$ *of a* $\triangle$ *is greater than either opposite interior* $\angle$.)

Also $\angle BAC > \angle XAC$. Ax. 11

(*For* $\angle XAC$ *is a part of* $\angle BAC$.)

Substituting in this inequality for $\angle XAC$ its equal, $\angle CXA$, we have the inequality

$\angle BAC > \angle CXA$. Ax. 9

Since $\angle BAC > \angle CXA$,

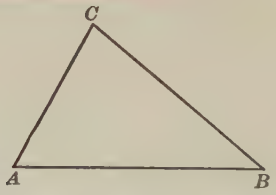
and $\angle CXA > \angle B$,

$\therefore \angle BAC > \angle B$. Ax. 10

(*If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.*) Q. E. D

PROPOSITION XXII. THEOREM

114. *If two angles of a triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater.*



Given the triangle ABC, with the angle A greater than the angle B.

To prove that $BC > CA$.

Proof. Now BC is either equal to CA, or less than CA, or greater than CA.

But if BC were equal to CA,

then the $\angle A$ would be equal to the $\angle B$. § 74

(*For they would be* $\triangle$ *opposite equal sides.*)

And if CA were greater than BC,

then the $\angle B$ would be greater than the $\angle A$. § 113

But if CA is not greater than BC, this is only another way of saying that BC is not less than CA.

We have, therefore, two conclusions to be considered,

$$\angle A = \angle B,$$

and $$\angle A < \angle B.$$

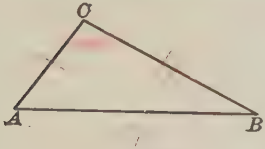
Both these conclusions are contrary to the given fact that the $\angle A$ is greater than the $\angle B$.

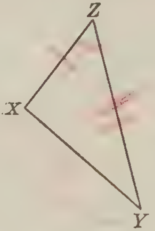
Since BC cannot be equal to CA or less than CA without violating the given condition, $\therefore BC > CA$. Q.E.D.

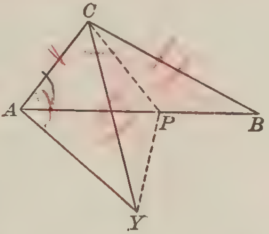
This proposition is the converse of Prop. XXI.

PROPOSITION XXIII. THEOREM

115. *If two triangles have two sides of the one equal respectively to two sides of the other, but the included angle of the first triangle greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*







Given the triangles *ABC* and *XYZ*, with *CA* equal to *ZX* and *BC* equal to *YZ*, but with the angle *C* greater than the angle *Z*.

To prove that $AB > XY.$

Proof. Place the $\triangle$ so that Z coincides with C and ZX falls along CA. Then X falls on A, since ZX is given equal to CA, and ZY falls within $\angle ACB$, since $\angle C$ is given greater than $\angle Z$.

Suppose CP drawn to bisect the $\angle YCB$, and draw YP.

Then since $CP = CP,$ Iden.

$CY = CB,$ Given

and $\angle YCP = \angle PCB,$ Hyp.

$\therefore \triangle PYC$ is congruent to $\triangle PBC.$ § 68

$\therefore PY = PB.$ § 67

Now $AP + PY > AY.$ Post. 3

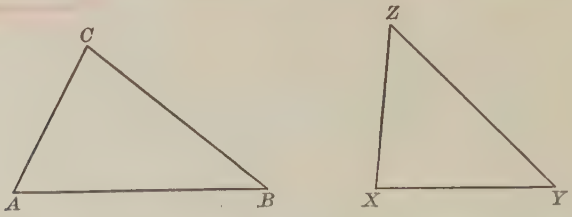
$\therefore AP + PB > AY.$ Ax. 9

$\therefore AB > AY.$ Ax. 11

$\therefore AB > XY$, by Ax. 9. Q. E. D.

PROPOSITION XXIV. THEOREM

116. *If two triangles have two sides of the one equal respectively to two sides of the other, but the third side of the first triangle greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*



Given the triangles ABC and XYZ, with CA equal to ZX and BC equal to YZ, but with AB greater than XY.

To prove that the $\angle C$ is greater than the $\angle Z$.

Proof. Now the $\angle C$ is either equal to the $\angle Z$, or less than the $\angle Z$, or greater than the $\angle Z$.

But if the $\angle C$ were equal to the $\angle Z$,

then the $\triangle ABC$ would be congruent to the $\triangle XYZ$, § 68

(*For it would have two sides and the included $\angle$ of the one equal respectively to two sides and the included $\angle$ of the other.*)

and AB would be equal to XY. § 67

And if the $\angle C$ were less than the $\angle Z$,

then AB would be less than XY. § 115

Both these conclusions are contrary to the given fact that AB is greater than XY.

$\therefore \angle C > \angle Z$. Q.E.D.

This proposition is the converse of Prop. XXIII.

117. Quadrilateral. A portion of a plane bounded by four straight lines is called a *quadrilateral.* ← 四角形

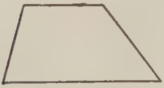
118. Kinds of Quadrilaterals. A quadrilateral may be

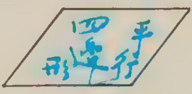
a *trapezoid,* having two sides parallel;
a *parallelogram,* having the opposite sides parallel.

四邊形

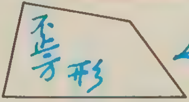
If the nonparallel sides are equal, a trapezoid is called *isosceles.*

A quadrilateral with no two sides parallel is called a *trapezium.*


Trapezoid



Parallelogram



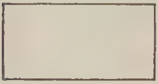
Trapezium

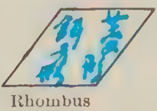
← 不平行四邊形

119. Kinds of Parallelograms. A parallelogram may be

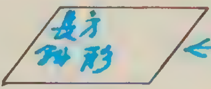
a *rectangle,* having its angles all right angles;
a *rhombus,* having its sides all equal.

A parallelogram with all its angles oblique is called a *rhomboid.*


Rectangle



Rhombus



Rhomboid

← 長菱形

120. Base. The side upon which a figure is supposed to rest is called the *base.*

If a quadrilateral has a side parallel to the base, this is called the *upper base,* the other being called the *lower base.*

In an isosceles triangle the vertex formed by the equal sides is taken as the *vertex* of the triangle, and the side opposite this vertex is taken as the *base* of the triangle.

121. Altitude. The perpendicular distance between the bases of a parallelogram or trapezoid is called the *altitude.*

The perpendicular distance from the vertex of a triangle to the base is called the *altitude* of the triangle.

122. Diagonal. The straight line joining two nonconsecutive vertices of any figure is called a *diagonal.* 不連續

PROPOSITION XXV. THEOREM

123. *Two angles whose sides are parallel each to each are either equal or supplementary.*

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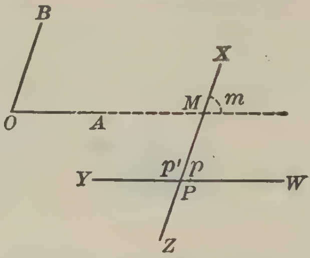


Given the angle *AOB* and the lines *WY* and *XZ* parallel to the sides and intersecting at *P*, the figure being lettered as shown.

To prove that $\angle p = \angle O$, *and that* $\angle p'$ *is supplementary to* $\angle O$.

Proof. Let OA meet XZ at M. Then in the figure

$\angle O = \angle m$, and $\angle p = \angle m$. § 102

(*If two* ∥ *lines are cut by a transversal, the ext.-int.* $\angle$s *are equal.*)

$\therefore \angle p = \angle O$. Ax. 8

Also $\angle p'$ is the supplement of $\angle p$. § 42

$\therefore \angle p'$ is supplementary to $\angle O$, by § 58. Q.E.D.

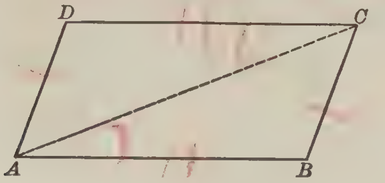
If the sides of two angles are parallel each to each, under what circumstances are the angles equal, and under what circumstances are they supplementary?

124. COROLLARY. *The opposite angles of a parallelogram are equal, and any two consecutive angles are supplementary.*

Draw the figure and explain how it is known that any angle is the supplement of its consecutive angle. If two opposite angles are supplements of the same angle, show that § 58 applies.

Proposition XXVI. Theorem

125. *The opposite sides of a parallelogram are equal.*



Given the parallelogram $ABCD$.

To prove that $BC = AD$, *and* $AB = DC$.

Proof. Draw the diagonal AC.

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In the $\triangle$s ABC and CDA,

$AC = AC$, Iden.

$\angle BAC = \angle DCA$,

and $\angle ACB = \angle CAD$. § 100

$\therefore \triangle ABC$ is congruent to $\triangle CDA$. § 72

$\therefore BC = AD$, and $AB = DC$, by § 67. Q.E.D.

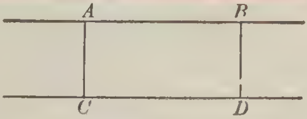
126. Corollary 1. *A diagonal divides a parallelogram into two congruent triangles.*

Upon what theorem does this depend?

127. Corollary 2. *Segments of parallel lines cut off by parallel lines are equal.*

How does this follow from the proposition?

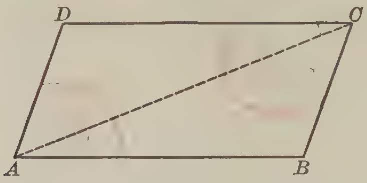
128. Corollary 3. *Two parallel lines are everywhere equally distant from each other.*



If AB and CD are parallel, what can be said of ⊥s dropped from *any* points in AB to CD (§ 127)? Hence what may be said of *all* points in AB with respect to their distance from CD?

PROPOSITION XXVII. THEOREM

129. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given the quadrilateral $ABCD$, having BC equal to AD, and AB equal to DC.

To prove that the quadrilateral $ABCD$ is a parallelogram.

Proof. Draw the diagonal AC.

In the $\triangle$s ABC and CDA,

$BC = AD$, Given

$AB = DC$, Given

and $AC = AC$. Iden.

$\therefore \triangle ABC$ is congruent to $\triangle CDA$. § 80

(*Two* $\triangle$s *are congruent if the three sides of the one are equal respectively to the three sides of the other.*)

$\therefore \angle BAC = \angle DCA$,

and $\angle ACB = \angle CAD$. § 67

$\therefore AB$ is $\parallel$ to DC,

and BC is $\parallel$ to AD. § 101

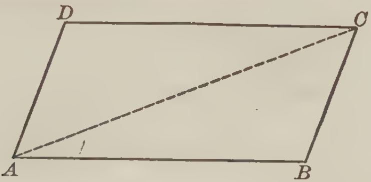
(*When two lines in the same plane are cut by a transversal, if the alt.-int.* $\angle$s *are equal, the two lines are* $\parallel$.)

$\therefore$ the quadrilateral $ABCD$ is a ▱, by § 118. Q.E.D.

This proposition is the converse of Prop. XXVI.

PROPOSITION XXVIII. THEOREM

130. *If two sides of a quadrilateral are equal and parallel, then the other two sides are equal and parallel, and the figure is a parallelogram.*



Given the quadrilateral *ABCD*, having *AB* equal and parallel to *DC*.

To prove that the quadrilateral ABCD is a parallelogram.

Proof. Draw the diagonal AC.

In the $\triangle$ ABC and CDA,

$AC = AC$, Iden.

$AB = DC$, Given

and $\angle BAC = \angle DCA$. § 100

(*If two ∥ lines are cut by a transversal, the alt.-int. ∠s are equal.*)

$\therefore \triangle ABC$ is congruent to $\triangle CDA$. § 68

(*Two △s are congruent if two sides and the included ∠ of the one are equal respectively to two sides and the included ∠ of the other.*)

$\therefore BC = AD$,

and $\angle ACB = \angle CAD$. § 67

$\therefore BC$ is $\parallel$ to AD. § 101

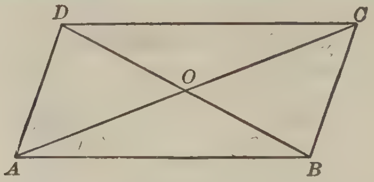
(*When two lines in the same plane are cut by a transversal, if the alt.-int. ∠s are equal, the two lines are ∥.*)

But AB is $\parallel$ to DC. Given

$\therefore$ the quadrilateral $ABCD$ is a ▱, by § 118. Q. E. D.

PROPOSITION XXIX. THEOREM

131. *The diagonals of a parallelogram bisect each other.*



Given the parallelogram *ABCD*, with the diagonals *AC* and *BD* intersecting at *O*.

To prove that $AO = OC$,

and $BO = OD$.

Proof. If we can show that the $\triangle ABO$ is congruent to the $\triangle CDO$, or that the $\triangle BCO$ is congruent to the $\triangle DAO$, the proposition is evidently proved, since the corresponding sides of the congruent triangles will be equal.

Now in the $\triangle$s ABO and CDO,

$$AB = CD, \qquad \S\ 125$$

(*The opposite sides of a ▱ are equal.*)

$$\angle BAO = \angle DCO,$$

and $$\angle OBA = \angle ODC. \qquad \S\ 100$$

(*If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*)

$$\therefore \triangle ABO \text{ is congruent to } \triangle CDO. \qquad \S\ 72$$

(*Two $\triangle$s are congruent if two $\angle$s and the included side of the one are equal respectively to two $\angle$s and the included side of the other.*)

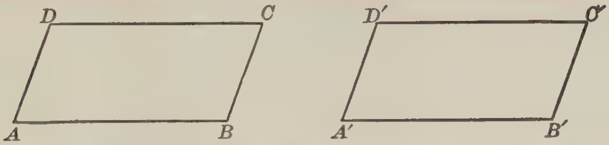
$$\therefore AO = OC,$$

and $$BO = OD. \qquad \S\ 67$$

(*Corresponding parts of congruent $\triangle$s are equal.*) Q.E.D.

PROPOSITION XXX. THEOREM

132. *Two parallelograms are congruent if two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.*



Given the parallelograms $ABCD$ and $A'B'C'D'$, with AB equal to $A'B'$, AD to $A'D'$, and angle A to angle A'.

To prove that the ▱s are congruent.

Proof. Place the ▱ $ABCD$ upon the ▱ $A'B'C'D'$ so that AB shall fall upon and coincide with its equal, $A'B'$. Post. 5

Then AD will fall along $A'D'$,

(For $\angle A$ is given equal to $\angle A'$.)

and D will fall on D'.

(For AD is given equal to $A'D'$.)

Now DC and $D'C'$ are both ∥ to $A'B'$ and are drawn through D'.

∴ DC will fall along $D'C'$. § 94

(Through a given point only one line can be drawn ∥ to a given line.)

Also BC and $B'C'$ are both ∥ to $A'D'$ and are drawn through B'.

∴ BC will fall along $B'C'$. § 94

∴ C will fall on C'. § 55

∴ the two ▱s coincide and are congruent, by § 66. Q.E.D.

133. COROLLARY. *Two rectangles having equal bases and equal altitudes are congruent.*

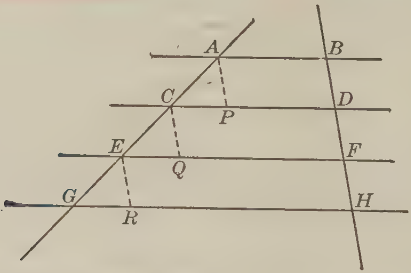
How is this shown to be a special case under the above proposition? What sides are equal, and what included angles are equal?

[illegible]

[illegible]

PROPOSITION XXXI. THEOREM

134. *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.*



Given the parallels AB, CD, EF, GH, intercepting equal segments BD, DF, FH on the transversal BH, and intercepting the segments AC, CE, EG on another transversal.

To prove that $AC = CE = EG$.

Proof. Suppose AP, CQ, and ER drawn ∥ to BH.

$\angle s$ APC, CQE, $ERG = \angle s$ BDC, DFE, FHG respectively. § 102

But $\angle s$ BDC, DFE, FHG are equal. § 102

$\therefore \angle s$ APC, CQE, ERG are equal. Ax. 8

AP, CQ, ER are parallel. § 96

Also $\angle s$ CAP, ECQ, GER are equal. § 102

Now $AP = BD$, $CQ = DF$, $ER = FH$. § 127

(*Segments of parallels cut off by parallels are equal.*)

But $BD = DF = FH$. Given

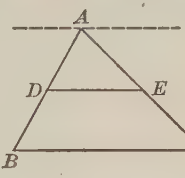
$\therefore AP = CQ = ER$. Ax. 8

$\therefore \triangle s$ CPA, EQC, and GRE are congruent. § 72

$\therefore AC = CE = EG$, by § 67. Q.E.D.

135. COROLLARY 1. *If a line is parallel to one side of a triangle and bisects another side, it bisects the third side also.*

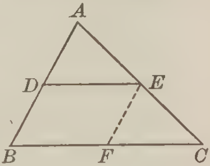
Let DE be ∥ to BC and bisect AB. Suppose a line is drawn through A ∥ to BC. Then how do we know this line to be ∥ to DE? Since it is given that the three ∥s intercept equal segments on the transversal AB, what can we say of the intercepted segments on AC? What can we then say that DE does to AC?



Write the proof of this corollary in full.

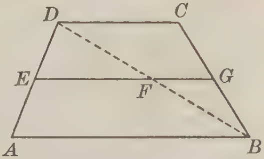
136. COROLLARY 2. *The line which joins the mid-points of two sides of a triangle is parallel to the third side, and is equal to half the third side.*

A line DE drawn through the mid-point of AB, ∥ to BC, divides AC in what way (§ 135)? Therefore the line joining the mid-points of AB and AC coincides with this parallel and is ∥ to BC. Also since EF drawn ∥ to AB bisects AC, how does it divide BC? What does this prove as to the relation of BF, FC, and BC? Since $BFED$ is a ▱ (§ 118), what do we know as to the equality of DE, BF, and $\frac{1}{2}BC$?



Write the proof of this corollary in full.

137. COROLLARY 3. *The line joining the mid-points of the nonparallel sides of a trapezoid is parallel to the bases and is equal to half the sum of the bases.*



Draw the diagonal DB. In the $\triangle ABD$ join E, the mid-point of AD, to F, the mid-point of DB. Then, by § 136, what relations exist between EF and AB? In the $\triangle DBC$ join F to G, the mid-point of BC. Then what relations exist between FG and DC? Since this relation exists, what relation exists between AB and FG? But only one line can be drawn through F ∥ to AB (§ 94). Therefore FG is the prolongation of EF. Hence EFG is parallel to AB and CD, and equal to $\frac{1}{2}(AB + DC)$.

Write the proof of this corollary in full.

138. Polygon. A portion of a plane bounded by a broken line is called a *polygon.*

The terms *sides*, *perimeter*, *angles*, *vertices*, and *diagonals* are employed in the usual sense in connection with polygons in general.

139. Polygons classified as to Sides. A polygon is

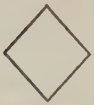
a *triangle,* if it has three sides;
a *quadrilateral,* if it has four sides;
a *pentagon,* if it has five sides;
a *hexagon,* if it has six sides.

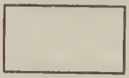
These names are sufficient for most cases. The next few names in order are *heptagon*, *octagon*, *nonagon*, *decagon*, *undecagon*, *dodecagon.*

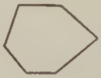
A polygon is *equilateral,* if all of its sides are equal.

140. Polygons classified as to Angles. A polygon is

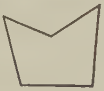
equiangular, if all of its angles are equal;
convex, if each of its angles is less than a straight angle;
concave, if it has an angle greater than a straight angle.


Equilateral


Equiangular


Hexagon

Convex


Concave

An angle of a polygon greater than a straight angle is called a *reëntrant angle.* When the term *polygon* is used, a convex polygon is understood.

141. Regular Polygon. A polygon that is both equiangular and equilateral is called a *regular polygon.*

142. Relation of Two Polygons. Two polygons are

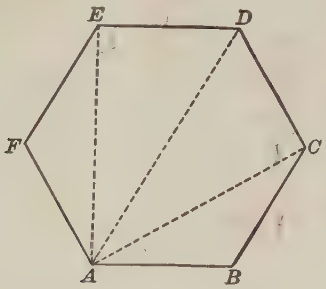
mutually equiangular, if the angles of the one are equal to the angles of the other respectively, taken in the same order;

mutually equilateral, if the sides of the one are equal to the sides of the other respectively, taken in the same order;

congruent, if mutually equiangular and mutually equilateral, since they then can be made to coincide.

PROPOSITION XXXII. THEOREM

143. *The sum of the interior angles of a polygon is equal to two right angles, taken as many times less two as the figure has sides.*



Given the polygon *ABCDEF*, having *n* sides.

To prove that the sum of the interior $\angle s = (n-2)\,2$ *rt.* $\angle s$.

Proof. From A draw the diagonals AC, AD, AE.

The sum of the $\angle s$ of the $\triangle s$ is equal to the sum of the $\angle s$ of the polygon. Ax. 11

Now there are $(n-2)$ $\triangle s$.

(*For there is one* $\triangle$ *for each side except the two sides adjacent to* A.)

The sum of the $\angle s$ of each $\triangle = 2$ rt. $\angle s$. § 107

$\therefore$ the sum of the $\angle s$ of the $(n-2)$ $\triangle s$, that is, the sum of the $\angle s$ of the polygon, is equal to $(n-2)\,2$ rt. $\angle s$, by Ax. 3. Q.E.D.

144. COROLLARY 1. *The sum of the angles of a quadrilateral equals four right angles; and if the angles are all equal, each is a right angle.*

145. COROLLARY 2. *Each angle of a regular polygon of n sides is equal to* $\dfrac{2(n-2)}{n}$ *right angles.*

七角 八角 五角 十角

EXERCISE 12

六角

1. What is the sum of the angles of (*a*) a pentagon? (*b*) a hexagon? (*c*) a heptagon? (*d*) an octagon? (*e*) a decagon? (*f*) a dodecagon? (*g*) a polygon of 24 sides?

十二角

2. What is the size of each angle of (*a*) a regular pentagon? (*b*) a regular hexagon? (*c*) a regular octagon? (*d*) a regular decagon? (*e*) a regular polygon of 32 sides?

3. How many sides has a regular polygon, each angle of which is $1\frac{3}{4}$ right angles? 32

4. How many sides has a regular polygon, each angle of which is $1\frac{3}{7}$ right angles?

5. How many sides has a regular polygon, each angle of which is 108°?

6. How many sides has a regular polygon, each angle of which is 140°?

7. How many sides has a regular polygon, each angle of which is 156°?

8. Four of the angles of a pentagon are 120°, 80°, 90°, and 100° respectively. Find the fifth angle.

9. Five of the angles of a hexagon are 100°, 120°, 130°, 150°, and 90° respectively. Find the sixth angle.

10. The angles of a quadrilateral are x, $2x$, $2x$, and $3x$. How many degrees are there in each?

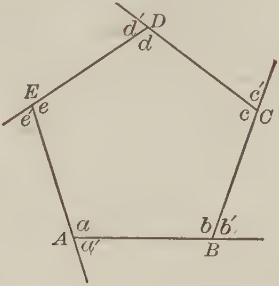
11. The angles of a quadrilateral are so related that the second is twice the first, the third three times the first, and the fourth four times the first. How many degrees in each?

12. The angles of a hexagon are x, $2\frac{1}{2}x$, $3\frac{1}{2}x$, $2x$, $2x$, and x. How many degrees are there in each?

13. The sum of two angles of a triangle is 100° and their difference is 40°. How many degrees are there in each of the three angles of the triangle?

PROPOSITION XXXIII. THEOREM

146. *The sum of the exterior angles of a polygon, made by producing each of its sides in succession, is equal to four right angles.*



Given the polygon *ABCDE*, having its *n* sides produced in succession.

To prove that the sum of the exterior $\angle s = 4$ *rt.* $\angle s$.

Proof. Denote the interior $\angle s$ of the polygon by a, b, c, d, e, and the corresponding exterior $\angle s$ by a', b', c', d', e'.

Then, considering each pair of adjacent angles,

$$\angle a + \angle a' = \text{a st. } \angle,$$

and $$\angle b + \angle b' = \text{a st. } \angle. \qquad \S\ 43$$

(*The two adjacent* $\angle s$ *which one straight line makes with another are together equal to a straight* $\angle$.)

In like manner, each pair of adj. $\angle s = \text{a st. } \angle$.

But the polygon has n sides and n angles.

Therefore the sum of the interior and exterior $\angle s$ is equal to n st. $\angle s$, or $2n$ rt. $\angle s$. Ax. 3

But the sum of the interior $\angle s = (n-2)\,2$ rt. $\angle s$ § 143

$$= 2n \text{ rt. } \angle s - 4 \text{ rt. } \angle s.$$

$\therefore$ the sum of the exterior $\angle s = 4$ rt. $\angle s$, by Ax. 2. Q.E.D.

EXERCISE 13

1. An exterior angle of a triangle is 130° and one of the opposite interior angles is 52°. Find the number of degrees in each angle of the triangle.

2. Two consecutive angles of a rectangle are bisected by lines meeting at P. How many degrees in the angle P?

3. Two angles of an equilateral triangle are bisected by lines meeting at P. How many degrees in the angle P?

4. The two base angles of an isosceles triangle are bisected by lines meeting at P. The vertical angle of the triangle is 30°. How many degrees in the angle P?

5. The vertical angle of an isosceles triangle is 40°. This and one of the base angles are bisected by lines meeting at P. How many degrees in the angle P?

6. One exterior angle of a parallelogram is one eighth of the sum of the four exterior angles. How many degrees in each angle of the parallelogram?

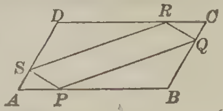
7. How many degrees in each exterior angle of a regular hexagon? of a regular octagon?

8. In a right triangle one acute angle is twice the other. How many degrees in each exterior angle of the triangle?

9. Make out a table showing the number of degrees in each interior angle and each exterior angle of regular polygons of three, four, five, · · ·, ten sides.

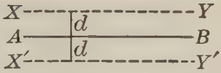
10. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

11. In this parallelogram $ABCD$, $AP = CR$, and $BQ = DS$. Prove that $PQRS$ is also a parallelogram.

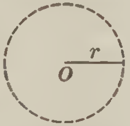


12. If the mid-points of the sides of a parallelogram are connected in order, the resulting figure is also a parallelogram.

147. Locus. The path of a point that moves in accordance with certain given geometric conditions is called the *locus* of the point.



Thus, considering only figures in a plane, a point at a given distance from a given line of indefinite length is evidently in one of two lines parallel to the given line and at the given distance from it. Thus, if AB is the given line and d the given distance, the locus is evidently the pair of parallel lines XY and $X'Y'$.



The locus of a point in a plane at a given distance r from a given point O is evidently the circle described about O as a center with a radius r.

The plural of *locus* (a Latin word meaning "place") is *loci* (pronounced lō-sī).

We may think of the locus as the *place* of all points that satisfy certain given geometric conditions, and speak of the locus of points. Both expressions, *locus of a point* and *locus of points*, are used in mathematics.

EXERCISE 14

State without proof the following loci in a plane:

1. The locus of a point 2 in. from a fixed point O.

2. The locus of the tip of the minute hand of a watch.

3. The locus of the center of the hub of a carriage wheel moving straight ahead on a level road.

4. The locus of a point 1 in. from each of two parallel lines that are 2 in. apart.

5. The locus of a point on this page and 1 in. from the edge.

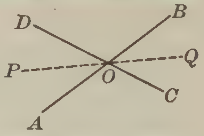
6. The locus of the point of a round lead pencil as it rolls along a desk.

7. The locus of the tips of a pair of shears as they open, provided the fulcrum (bolt or screw) remains always fixed in one position.

8. The locus of the center of a circle that rolls around another circle, always just touching it.

148. Proof of a Locus. To prove that a certain line or group of lines is the locus of a point that fulfills a given condition, it is necessary and sufficient to prove two things:

1. *That any point in the supposed locus satisfies the condition.*
2. *That any point outside the supposed locus does not satisfy the given condition.*



For example, if we wish to find the locus of a point equidistant from these intersecting lines AB, CD, it is not sufficient to prove that any point on the angle-bisector PQ is equidistant from AB and CD, because this may be only part of the locus. It is necessary to prove that no point outside of PQ satisfies the condition. In fact, in this case there is another line in the locus, the bisector of the $\angle BOD$, as will be shown in § 152.

149. Perpendicular Bisector. A line that bisects a given line and is perpendicular to it is called the *perpendicular bisector* of the line.

EXERCISE 15

Draw the following loci, giving no proofs:

1. The locus of a point $\frac{1}{4}$ in. below the base of a given triangle ABC.

2. The locus of a point $\frac{1}{2}$ in. from a given line AB.

3. The locus of a point 1 in. from a given point O.

4. The locus of a point $\frac{1}{2}$ in. outside the circle described about a given point O with a radius $1\frac{1}{2}$ in.

5. The locus of a point $\frac{1}{2}$ in. within the circle described about a given point O with a radius $1\frac{1}{2}$ in.

6. The locus of a point $\frac{1}{2}$ in. from the circle described about a given point O with a radius $1\frac{1}{2}$ in.

7. The locus of a point $\frac{1}{2}$ in. from each of two given parallel lines that are 1 in. apart.

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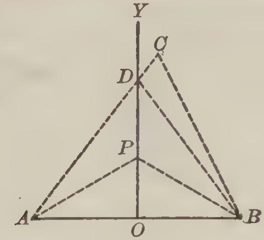
PROPOSITION XXXIV. THEOREM

150. *The locus of a point equidistant from the extremities of a given line is the perpendicular bisector of that line.*



Given ***YO*****, the perpendicular bisector of the line** ***AB*****.**

To prove that YO is the locus of a point equidistant from A and B.

Proof. Let P be any point in YO, and C any point not in YO.

Draw the lines PA, PB, CA, and CB.

Since $AO = BO$, Given

and $OP = OP$, Iden.

$\therefore$ rt. $\triangle AOP$ is congruent to rt. $\triangle BOP$. § 90

$\therefore PA = PB$. § 67

Let CA cut the $\perp$ at D, and draw DB.

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Then, as above, $DA = DB$.

But $CB < CD + DB$. Post. 3

$\therefore CB < CD + DA$. Ax. 9

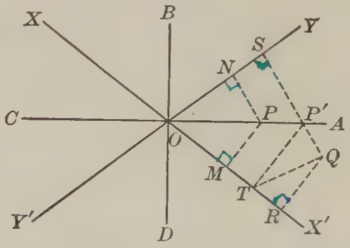
$\therefore CB < CA$. Ax. 11

$\therefore YO$ is the required locus, by § 148. Q.E.D.

151. COROLLARY. *Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.*

PROPOSITION XXXV. THEOREM

152. *The locus of a point equidistant from two given intersecting lines is a pair of lines bisecting the angles formed by those lines.*



Given XX' **and** YY' **intersecting at** O, AC **the bisector of angle** $X'OY$, **and** BD **the bisector of angle** YOX.

To prove that the pair of lines AC *and* BD *is the locus of a point equidistant from* XX' *and* YY'.

Proof. Let P be any point on AC or BD, and Q any point not on AC or BD. Let PM and QR be $\perp$ to XX', PN and QS to YY'.

Since $\quad \angle MOP = \angle PON,\quad$ Given

and $\quad OP = OP,\quad$ Iden.

$\therefore$ rt. $\triangle OMP$ is congruent to rt. $\triangle ONP$. § 91

$\therefore PM = PN.$ § 67

Let QS cut AO at P'. Draw $P'T \perp$ to XX', and draw QT.

Then, as above, $\quad P'T = P'S.$

But $\quad P'T + P'Q > QT,\quad$ Post. 3

and $\quad QT > QR.\quad$ § 86

$\therefore P'T + P'Q > QR.\quad$ Ax. 10

Substituting, $\quad P'S + P'Q > QR$, or $QS > QR.\quad$ Ax. 9

$\therefore$ the pair of lines is the required locus, by § 148. Q.E.D.

綜合的證明方法

153. The Synthetic Method of Proof. The method of proof in which known truths are put together in order to obtain a new truth is called the *synthetic method.*

This is the method used in most of the theorems already given. The proposition usually suggests some known propositions already proved, and from these we proceed to the proof required. The exercises on this page and on pages 78 and 79 may be proved by the synthetic method.

共同一点之線

154. Concurrent Lines. If two or more lines pass through the same point, they are called *concurrent lines.*

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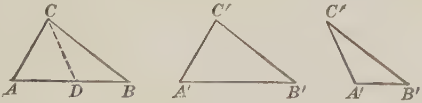
155. Median. A line from any vertex of a triangle to the mid-point of the opposite side is called a *median* of the triangle.

EXERCISE 16

1. If two triangles have two sides of the one equal respectively to two sides of the other, and the angles opposite two equal sides equal, the angles opposite the other two equal sides are equal or supplementary, and if equal the triangles are congruent.

Let $AC = A'C'$, $BC = B'C'$, and $\angle B = \angle B'$.

Place $\triangle A'B'C'$ on $\triangle ABC$ so that $B'C'$ shall coincide with BC, and $\angle A'$ and $\angle A$ shall be on the same side of BC.



Since $\angle B' = \angle B$, $B'A'$ will fall along what line? Then A' will fall at A or at some other point in BA, as D. If A' falls at A, what do we know about the congruency of the $\triangle$ $A'B'C'$ and ABC?

If A' falls at D, what about the congruency of the $\triangle$ $A'B'C'$ and DBC?

Since $CD = C'A' = CA$, what about the relation of $\angle A$ to $\angle CDA$?

Then what about the relation of the $\angle$ CDA and BDC?

Then what about the relation of the $\angle$ A and BDC?

Draw figures and show that the triangles are congruent:

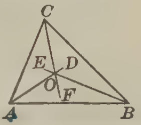
1. If the given angles B and B' are both right or both obtuse.
2. If the angles A and A' are both acute, both right, or both obtuse.
3. If AC and $A'C'$ are not less than BC and $B'C'$ respectively.

[illegible]

2. The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides of the triangle.

The bisectors of two angles, as AD and BE, intersect as at O. Why? Now show that O is equidistant from AC and AB, also from BC and AB, and hence from AC and BC. Therefore, where does O lie with respect to the bisector CF?

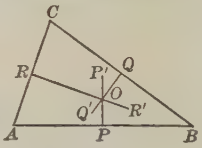
This point O is called the *incenter* of the triangle.



3. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.

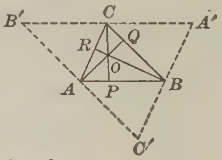
The ⊥ bisectors of two sides, as QQ' and RR', intersect as at O. Why? Now show that O is equidistant from B and C, also from C and A, and hence from A and B. Therefore, where does O lie with respect to the ⊥ bisector PP'?

This point O is called the *circumcenter* of the triangle.



4. The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

Let the ⊥s be AQ, BR, and CP. Through A, B, C suppose $B'C'$, $C'A'$, and $A'B'$ drawn ∥ to CB, AC, and BA respectively. Now show that $C'A = BC = AB'$. In the same way, what are the mid-points of $C'A'$ and $A'B'$? How does this prove that AQ, BR, and CP are the ⊥ bisectors of the sides of the △ $A'B'C'$? Proceed as in Ex. 3.

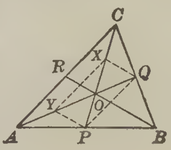


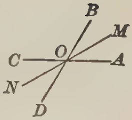
This point O is called the *orthocenter* of the triangle.

5. The medians of a triangle are concurrent in a point two thirds of the distance from each vertex to the middle of the opposite side.

Two medians, as AQ and CP, meet as at O. If Y is the mid-point of AO, and X of CO, show that YX and PQ are ∥ to AC and equal to $\frac{1}{2}AC$. Then show that $AY = YO = OQ$, and $CX = XO = OP$. Hence *any* median cuts off on *any* other median what part of the distance from the vertex to the mid-point of the opposite side?

This point O is called the *centroid* of the triangle.



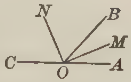


6. The bisectors of two vertical angles are in the same straight line.

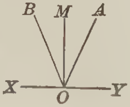
7. The bisector of one of two vertical angles bisects the other.

8. The bisectors of two supplementary adjacent angles are perpendicular to each other.

9. The bisectors of the two pairs of vertical angles formed by two intersecting lines are perpendicular to each other.

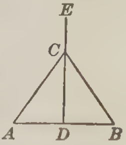


10. If the bisectors of two adjacent angles are perpendicular to each other, the adjacent angles are supplementary.



11. If an angle is bisected, and if a line is drawn through the vertex perpendicular to the bisector, this line forms equal angles with the sides of the given angle.

12. The bisector of the vertical angle of an isosceles triangle bisects the base and is perpendicular to the base.



13. The perpendicular bisector of the base of an isosceles triangle passes through the vertex and bisects the angle at the vertex.

14. If the perpendicular bisector of the base of a triangle passes through the vertex, the triangle is isosceles.

15. Any point in the bisector of the vertical angle of an isosceles triangle is equidistant from the extremities of the base.

16. If two isosceles triangles are on the same base, a line passing through their vertices is perpendicular to the base and bisects the base.

17. Two angles whose sides are perpendicular each to each are either equal or supplementary.

Under what circumstances are the angles equal, and under what circumstances are they supplementary?

156. The Analytic Method of Proof. The method of proof that asserts that a proposition under consideration is true if another proposition is true, and so on, step by step, until a known truth is reached, is called the *analytic method*.

This is the method resorted to when we do not see how to start the ordinary synthetic proof. The exercises on this page and on pages 81 and 82 may be investigated by the analytic method.

弦 直角三角形之斜

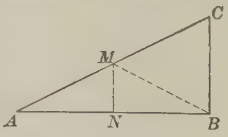
EXERCISE 17

1. The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.

Given M the mid-point of AC, the hypotenuse of the rt. $\triangle ABC$.

To prove that M is equidistant from A, B, and C.

We may reason thus: M is equidistant from A, B, and C if $AM = BM$. Why is this the case?



$AM = BM$ if the $\perp MN$ cuts $\triangle ABM$ into two congruent $\triangle$s.

$\triangle ANM$ is congruent to $\triangle BNM$ if $AN = NB$.

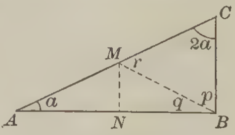
But AN does equal NB (§ 135), because MN is $\parallel$ to CB, and $AM = MC$.

Therefore the proposition is true.

We may now, in writing our proof, begin with this last step and work backwards, as in the synthetic proofs already considered.

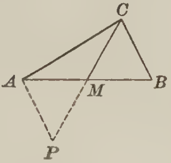
2. If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter side.

Given $\angle A = \angle a$, and $\angle C = \angle 2a$, to prove that AC is double BC. Let M be the mid-point of AC. Then AC is double BC if $AM = BC$. Why? Now if we draw $MN \parallel$ to CB, what can be said of the relation of AN and NB? Why? Then what may be said of $\triangle$s ANM and BNM? Why? Then what may be said of AM and BM? of $\angle a$ and $\angle q$? Therefore the proposition is true if $BM = BC$. But $BM = BC$ if $\angle 2a = \angle r$, or if $\angle 2a = \angle a + \angle q$, or if $\angle a = \angle q$. But $\angle a = \angle q$ because we have proved that $AM = BM$.



Now reverse this reasoning and write the proof in the usual synthetic form.

3. A median of a triangle is less than half the sum of the two adjacent sides.



Given CM a median of the $\triangle ABC$.

To prove that $CM < \frac{1}{2}(BC + CA)$.

Now $CM < \frac{1}{2}(BC + CA)$,

if $2CM < BC + CA$.

This suggests producing CM by its own length to P, and drawing AP.

Then $CP = 2CM$,

and $2CM < BC + CA$ if $CP < BC + CA$.

But $CP < AP + CA$. Post. 3

$\therefore CP < BC + CA$ if $BC = AP$,

and $BC = AP$ if $\triangle MBC$ is congruent to $\triangle MAP$. § 67

But $\triangle MBC$ is congruent to $\triangle MAP$, § 68

for $MB = MA$, Given

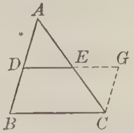
$CM = MP$, Hyp.

and $\angle BMC = \angle AMP$. § 60

$\therefore CP < BC + CA$.

$\therefore CM < \frac{1}{2}(BC + CA)$.

4. The line which bisects two sides of a triangle is parallel to the third side.



Given AD equal to DB, and AE equal to EC.

To prove that DE is $\parallel$ to BC.

Suppose a line drawn from C $\parallel$ to BA, and suppose DE produced to meet it at G.

DE is $\parallel$ to BC if $BCGD$ is a ▱. § 118

$BCGD$ is a ▱ if $CG = BD$. § 130

$CG = BD$ if each is equal to AD. Ax. 8

Now $BD = AD$, Given

and $CG = AD$ if $\triangle CGE$ is congruent to $\triangle ADE$. § 67

But $\triangle CGE$ is congruent to $\triangle ADE$, § 72

for $EC = AE$, Given

$\angle CEG = \angle AED$, § 60

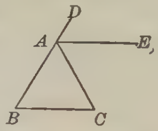
and $\angle GCE = \angle A$. § 100

5. Two isosceles triangles are congruent if a side and an angle of the one are equal respectively to the corresponding side and angle of the other.

The $\triangle$ are congruent if what three corresponding parts are equal?

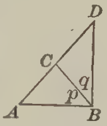
6. The bisector of an exterior angle of an isosceles triangle, formed by producing one of the equal sides through the vertex, is parallel to the base.

AE is ∥ to *BC* if what angles are equal? These angles are equal if $\angle CAD$ is twice what angle in the $\triangle$?



7. If one of the equal sides of an isosceles triangle is produced through the vertex by its own length, the line joining the end of the side produced to the nearer end of the base is perpendicular to the base.

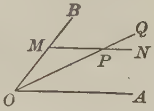
$\angle DBA$ is a rt. $\angle$ if it equals the sum of what $\angle$s of $\triangle ABD$? It equals this sum if $\angle p$ equals what angle and $\angle q$ equals what other angle?



8. If the equal sides of an isosceles triangle are produced through the vertex so that the external segments are equal, the extremities of these segments are equidistant from the extremities of the base respectively.

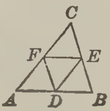
9. If the line drawn from the vertex of a triangle to the mid-point of the base is equal to half the base, the angle at the vertex is a right angle.

10. If through any point in the bisector of an angle a line is drawn parallel to either side of the angle, the triangle thus formed is isosceles.



11. Through any point *C* in the line *AB* an intersecting line is drawn, and from any two points in this line equidistant from *C* perpendiculars are drawn to *AB* or *AB* produced. Prove that these perpendiculars are equal.

12. The lines joining the mid-points of the sides of a triangle divide the triangle into four congruent triangles.



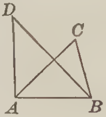
157. The Indirect Method of Proof. The method of proof that assumes the proposition false and then shows that this assumption is absurd is called the *indirect method* or the *reductio ad absurdum.*

This method forms a kind of last resort in the proof of a proposition, after the synthetic and analytic methods have failed.

EXERCISE 18

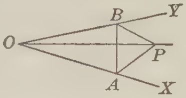
1. Given ABC and ABD, two triangles on the same base AB, and on the same side of it, the vertex of each triangle being outside the other triangle. Prove that if AC equals AD, then BC cannot equal BD.

Assume that $BC = BD$ and show that the result is absurd, since it would make D fall on C, which is contrary to the given conditions.



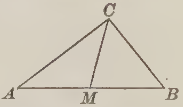
2. On the sides of the angle XOY two equal segments OA and OB are taken. On AB a triangle APB is constructed with AP greater than BP. Prove that OP cannot bisect the angle XOY.

Assume that OP does bisect $\angle XOY$. What is the result? Is this result possible?



3. From M, the mid-point of a line AB, MC is drawn oblique to AB. Prove that CA cannot equal CB.

Assume that CA does equal CB. What is the result? Is this result possible?



4. If perpendiculars are drawn to the sides of an acute angle from a point within the angle, they cannot inclose a right angle or an acute angle.

Assume that they inclose a right angle and show that this leads to an absurdity. Similarly for an acute angle.

5. One of the equal angles of an isosceles triangle is five ninths of a right angle. Prove that the angle at the vertex cannot be a right angle.

Assume that it is a right angle. Is the result possible?

158. General Suggestions for proving Theorems. The following general suggestions will often be helpful:

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1. *Draw the figures as accurately as possible.*

This is especially helpful at first. A proof is often rendered difficult simply because the figure is carelessly drawn. If one line is to be laid off equal to another, or if one angle is to be made equal to another, do this by the help of the compasses or by measuring with a ruler.

2. *Draw as general figures as possible.*

If you wish to prove a proposition about a triangle, take a scalene triangle. If an equilateral triangle, for example, is taken, it may lead to believing something true for every kind of a triangle, when, in fact, it is true for only that particular kind.

3. *After drawing the figure state very clearly exactly what is given and exactly what is to be proved.*

Many of the difficulties of geometry come from failing to keep in mind *exactly* what is given and *exactly* what is to be proved.

4. *Then proceed synthetically with the proof if you see how to begin. If you do not see how to begin, try the analytic method, stating clearly that you could prove this if you could prove that, and so on until you reach a known proposition.*

5. *If two lines are to be proved equal, try to prove them corresponding sides of congruent triangles, or sides of an isosceles triangle, or opposite sides of a parallelogram, or segments between parallels that cut equal segments from another transversal.*

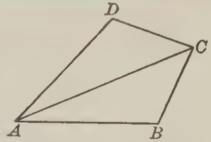
6. *If two angles are to be proved equal, try to prove them alternate-interior or exterior-interior angles of parallel lines, or corresponding angles of congruent triangles, or base angles of an isosceles triangle, or opposite angles of a parallelogram.*

7. *If one angle is to be proved greater than another, it is probably an exterior angle of a triangle, or an angle opposite the greater side of a triangle.*

8. *If one line is to be proved greater than another, it is probably opposite the greater angle of a triangle.*

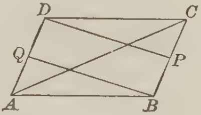
EXERCISE 19

Prove the following propositions referring to equal lines:

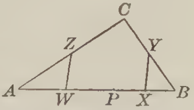


1. If the sides AB and AD of a quadrilateral $ABCD$ are equal, and if the diagonal AC bisects the angle at A, then BC is equal to DC.

2. A line is drawn terminated by two parallel lines. Through its mid-point any line is drawn terminated by the parallels. Prove that the second line is bisected by the first.



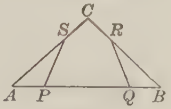
3. In a parallelogram $ABCD$ the line BQ bisects AD, and DP bisects BC. Prove that BQ and DP trisect AC.



4. On the base AB of a triangle ABC any point P is taken. The lines AP, PB, BC, and CA are bisected by W, X, Y, and Z respectively. Prove that XY is equal to WZ.

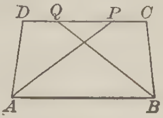
5. In an isosceles triangle the medians drawn to the equal sides are equal.

6. In the square $ABCD$, CD is bisected by Q, and P and R are taken on AB so that AP equals BR. Prove that PQ equals RQ.



7. In this figure $AC = BC$, and $AP = BQ = CR = CS$. Prove that $QR = PS$.

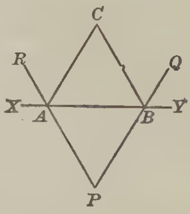
8. From the vertex and the mid-points of the equal sides of an isosceles triangle lines are drawn perpendicular to the base. Prove that they divide the base into four equal parts.



9. In the quadrilateral $ABCD$ it is known that AB is parallel to DC, and that angle C equals angle D. On CD two points are taken such that $CP = DQ$. Prove that $AP = BQ$.

EXERCISE 20

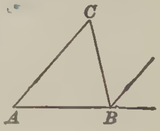
Prove the following propositions referring to equal angles:



1. In this figure it is given that $AC = BC$, and that BQ and AR bisect the angles YBC and CAX respectively. Prove that $\triangle APB$ is isosceles.

2. If through the vertices of an isosceles triangle lines are drawn parallel to the opposite sides, they form an isosceles triangle.

3. If the vertical angles of two isosceles triangles coincide, the bases either coincide or are parallel.

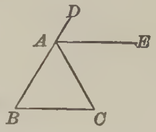


4. In which direction must the side of a triangle be produced so as to intersect the bisector of the opposite exterior angle?

Consider the cases, $\angle A < \angle C$, $\angle A = \angle C$, $\angle A > \angle C$.

5. The bisectors of the equal angles of an isosceles triangle form, together with the base, an isosceles triangle.

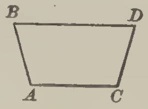
6. The bisectors of the base angles of an equilateral triangle form an angle equal to the exterior angle at the vertex of the triangle.



7. If the bisector of an exterior angle of a triangle is parallel to the opposite side, the triangle is isosceles.

8. A line drawn parallel to the base of an isosceles triangle makes equal angles with the sides or the sides produced.

9. A line drawn at right angles to AB, the base of an isosceles triangle ABC, cuts AC at P and BC produced at Q. Prove that PCQ is an isosceles triangle.



10. In this figure, if $AB = CD$, and $\angle A = \angle C$, then BD is parallel to AC.

EXERCISE 21

Prove the following propositions by showing that two triangles are congruent:

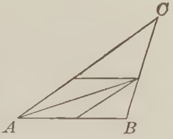
1. A perpendicular to the bisector of an angle forms with the sides an isosceles triangle.

2. If two lines bisect each other at right angles, any point in either is equidistant from the extremities of the other.

3. From B a perpendicular is drawn to the bisector of the angle A of the triangle ABC, meeting it at X, and meeting AC or AC produced at Y. Prove that $BX = XY$.

4. If through any point equally distant from two parallel lines two lines are drawn cutting the parallels, they intercept equal segments on these parallels.

5. If from the point where the bisector of an angle of a triangle meets the opposite side, parallels are drawn to the other two sides, and terminated by the sides, these parallels are equal.



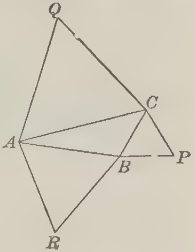
6. The diagonals of a square are perpendicular to each other and bisect the angles of the square.

7. If from a vertex of a square there are drawn line-segments to the mid-points of the two sides not adjacent to the vertex, these line-segments are equal.

8. If either diagonal of a parallelogram bisects one of the angles, the sides of the parallelogram are all equal.

9. On the sides of any triangle ABC equilateral triangles BPC, CQA, ARB are constructed. Prove that $AP = BQ = CR$.

How can we prove that $\triangle ABP$ is congruent to $\triangle RBC$? Also that $\triangle ARC$ is congruent to $\triangle ABQ$? Does this prove the proposition?



EXERCISE 22

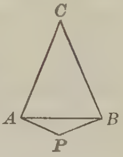
Prove the following propositions relating to the sum of the angles of a polygon:

1. An exterior angle of an acute triangle or of a right triangle cannot be acute.

2. If the sum of two angles of a triangle equals the third angle, the triangle is a right triangle.

3. If the line joining any vertex of a triangle to the mid-point of the opposite side divides the triangle into two isosceles triangles, the original triangle is a right triangle.

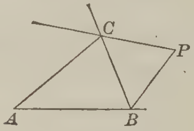
4. If the vertical angles of two isosceles triangles are supplements one of the other, the base angles of the one are complements of those of the other.



5. From the extremities of the base AB of a triangle ABC perpendiculars to the other two sides are drawn, meeting at P. Prove that the angle P is the supplement of the angle C.

6. If two sides of a quadrilateral are parallel, and the other two sides are equal but not parallel, the sums of the two pairs of opposite angles are equal.

7. The bisectors of two consecutive angles of a parallelogram are perpendicular to each other.



8. The exterior angles at B and C of any triangle ABC are bisected by lines meeting at P. Prove that the angle at P together with half the angle A equals a right angle.

9. The opposite angles of the quadrilateral formed by the bisectors of the interior angles of any quadrilateral are supplemental.

10. Show that Ex. 9 is true, if the bisectors of the exterior angles are taken.

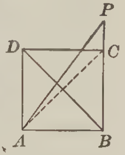
EXERCISE 23

Prove the following propositions referring to greater lines or greater angles:

1. In the triangle ABC the angle A is bisected by a line meeting BC at D. Prove that BA is greater than BD, and CA greater than CD.

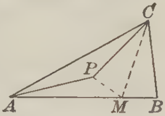
2. In the quadrilateral $ABCD$ it is known that AD is the longest side and BC the shortest side. Prove that the angle B is greater than the angle D, and the angle C greater than the angle A.

3. A line is drawn from the vertex A of a square $ABCD$ so as to cut CD and to meet BC produced in P. Prove that AP is greater than DB.



4. If the angle between two adjacent sides of a parallelogram is increased, the length of the sides remaining unchanged, the diagonal from the vertex of this angle is diminished.

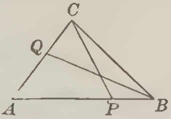
5. Within a triangle ABC a point P is taken such that $CP = CB$. Prove that AB is always greater than AP.



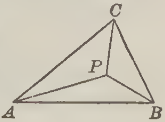
6. In a quadrilateral $ABCD$ it is known that AD equals BC and that the angle C is less than the angle D. Prove that the diagonal AC is greater than the diagonal BD.

7. In the quadrilateral $ABCD$ it is known that AD equals BC and that the angle D is greater than the angle C. Prove that the angle B is greater than the angle A.

8. In the triangle ABC the side AB is greater than AC. On AB and AC respectively BP is taken equal to CQ. Prove that BQ is greater than CP.



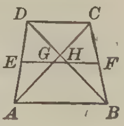
9. The sum of the distances of any point from the three vertices of a triangle is greater than half the sum of the sides.



EXERCISE 24

Prove the following miscellaneous exercises:

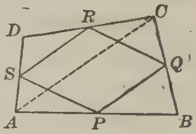
1. The line joining the mid-points of the nonparallel sides of a trapezoid passes through the mid-points of the two diagonals.



How is EF related to AB and DC? Why?

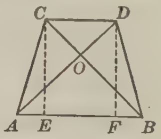
Since EF bisects BC and AD, how does it divide AC and BD? Why?

2. The lines joining the mid-points of the consecutive sides of any quadrilateral form a parallelogram.



How are PQ and SR related to AC?

3. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.



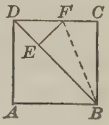
Draw CE and $DF \perp$ to AB.

How is $\triangle ADF$ related to $\triangle BCE$? Why?

Then how is $\angle FAD$ related to $\angle CBA$?

Then how is $\triangle ABC$ related to $\triangle BAD$? Why?

4. If from the diagonal DB, of a square $ABCD$, BE is cut off equal to BC, and EF is drawn perpendicular to BD, meeting DC at F, then DE is equal to EF and also to FC.

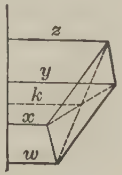


How many degrees in ∡ EDF and DFE? How is DE related to EF? Why?

Then how is rt. $\triangle BEF$ related to rt. $\triangle BCF$? Why?

5. If the opposite sides of a hexagon are equal and parallel, the diagonals that join opposite vertices meet in a point.

6. If perpendiculars are drawn from the four vertices of a parallelogram to any line outside the parallelogram, the sum of the perpendiculars from one pair of opposite vertices equals the sum of those from the other pair.



How are $x + y$ and $w + z$ related to k?

EXERCISE 25

EXAMINATION QUESTIONS

1. The sum of the four sides of any quadrilateral is greater than the sum of the diagonals.

2. The lines joining the mid-points of the sides of a square, taken in order, form a square.

3. In a quadrilateral the angle between the bisectors of two consecutive angles is one half the sum of the other two angles.

4. If the opposite sides of a hexagon are equal, does it follow that they are parallel? Give reasons for your answer.

5. In a triangle ABC the side BC is bisected at P and AB is bisected at Q. AP is produced to R so that $AP = PR$, and CQ is produced to S so that $CQ = QS$. Prove that S, B, and R are in a straight line.

6. If the diagonals of a parallelogram are equal, all of the angles of the parallelogram are equal.

7. In the triangle ABC, $\angle A = 60°$ and $\angle B > \angle C$. Which is the longest and which is the shortest side of the triangle? Prove it.

8. How many sides has a polygon each of whose interior angles is equal to 175°?

9. Given the quadrilateral $ABCD$, with AB equal to AD, and BC equal to CD. Prove that the diagonal AC bisects the angle DCB and is perpendicular to the diagonal BD.

10. In how many ways can two congruent triangles be put together to form a parallelogram? Draw the diagrams.

11. The sides of a polygon of an odd number of sides are produced to meet, thus forming a star-shaped figure. What is the sum of the angles at the points of the star?

The propositions in Exercise 25 are taken from recent college entrance examination papers.

EXERCISE 26

REVIEW QUESTIONS

1. Define and illustrate rectilinear and curvilinear figures.

2. Upon what does the size of an angle depend?

3. What is meant by the bisector of a magnitude? Illustrate when the magnitude is a line; an angle.

4. Define perpendicular and state three facts relating to a perpendicular to a line.

5. Name and define the parts of a triangle and such special lines connected with a triangle as you have thus far studied.

6. Classify angles.

7. Classify triangles as to angles; as to sides.

8. Define and illustrate complementary, supplementary, and conjugate angles.

9. What are the two classes of assumptions in geometry? Give the list of each.

10. State all of the conditions of congruency of two triangles.

11. What is meant by the converse of a proposition?

12. Are two triangles always congruent if three parts of the one are respectively equal to three parts of the other?

13. State three tests for determining whether one line is parallel to another.

14. State the proposition relating to the sum of the angles of a triangle, and state a proposition that can be proved by its use.

15. State a proposition relating to two unequal angles of a triangle; to two unequal sides of a triangle.

16. Must a triangle be equiangular if equilateral? Must a triangle be equilateral if equiangular?

17. Classify polygons as to sides; as to angles.

18. Define locus and give three illustrations.

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BOOK II

THE CIRCLE

159. Circle. A closed curve lying in a plane, and such that all of its points are equally distant from a fixed point in the plane, is called a *circle.*

160. Circle as a Locus. It follows that *the locus of a point in a plane at a given distance from a fixed point is a circle.*

161. Radius. A straight line from the center to the circle is called a *radius.*

162. Equal Radii. It follows that *all radii of the same circle or of equal circles are equal,* and that *all circles of equal radii are equal.*

163. Diameter. A straight line through the center, terminated at each end by the circle, is called a *diameter.*

Since a diameter equals two radii, it follows that *all diameters of the same circle or of equal circles are equal.*

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164. Arc. Any portion of a circle is called an *arc.*

An arc that is half of a circle is called a *semicircle.*

An arc less than a semicircle is called a *minor arc,* and an arc greater than a semicircle is called a *major arc.* The word *arc* taken alone is generally understood to mean a minor arc.

165. Central Angle. If the vertex of an angle is at the center of a circle and the sides are radii of the circle, the angle is called a *central angle.*

An angle is said to *intercept* any arc cut off by its sides, and the arc is said to *subtend* the angle.

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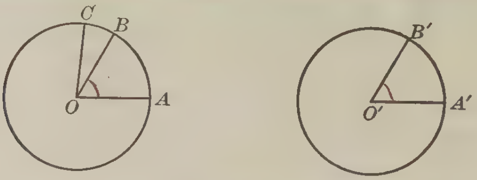
Proposition I. Theorem

166. *In the same circle or in equal circles equal central angles intercept equal arcs; and of two unequal central angles the greater intercepts the greater arc.*



Given two equal circles with centers O and O', with angles AOB and $A'O'B'$ equal, and with angle AOC greater than angle $A'O'B'$.

To prove that 1. *arc* $AB =$ *arc* $A'B'$;

2. *arc* $AC >$ *arc* $A'B'$.

Proof. 1. Place the circle with center O on the circle with center O' so that $\angle AOB$ shall coincide with its equal, $\angle A'O'B'$. In the case of the same circle, swing one angle about O until it coincides with its equal angle. Post. 5

Then A falls on A', and B on B'. § 162

(*Radii of equal circles are equal.*)

$\therefore$ arc AB coincides with arc $A'B'$. § 159

(*Every point of each is equally distant from the center.*)

Proof. 2. Since $\angle AOC$ is greater than $\angle A'O'B'$, Given

and $\angle AOB = \angle A'O'B'$, Given

therefore $\angle AOC$ is greater than $\angle AOB$. Ax. 9

Therefore OC lies outside $\angle AOB$.

$\therefore$ arc $AC >$ arc AB. Ax. 11

But arc $AB =$ arc $A'B'$.

$\therefore$ arc $AC >$ arc $A'B'$, by Ax. 9. Q.E.D.

PROPOSITION II. THEOREM

167. *In the same circle or in equal circles equal arcs subtend equal central angles; and of two unequal arcs the greater subtends the greater central angle.*

Given two equal circles with centers O and O', with arcs AB and $A'B'$ equal, and with arc AC greater than arc $A'B'$.

To prove that 1. $\angle AOB = \angle A'O'B'$;

2. $\angle AOC > \angle A'O'B'$.

Proof. 1. Using the figure of Prop. I, place the circle with center O on the circle with center O' so that OA shall fall on its equal $O'A'$, and the arc AB on its equal $A'B'$. Post. 5

Then OB coincides with $O'B'$. Post. 1

$\therefore \angle AOB = \angle A'O'B'$. § 23

Proof. 2. Since arc $AC >$ arc $A'B'$, it is greater than arc AB, the equal of arc $A'B'$, and OB lies within the $\angle AOC$. Ax. 9

$\therefore \angle AOC > \angle AOB$. Ax. 11

$\therefore \angle AOC > \angle A'O'B'$, by Ax. 9. Q.E.D.

This proposition is the converse of Prop. I.

168. Law of Converse Theorems. Of four magnitudes, a, b, x, y, if

(1) $a > b$ when $x > y$,

(2) $a = b$ when $x = y$,

and (3) $a < b$ when $x < y$,

then *the converses of these three statements are always true.*

For when $a > b$ it is impossible that $x = y$, for then a would equal b by (2); or that $x < y$, for then a would be less than b by (3). Hence $x > y$ when $a > b$. In the same way, $x = y$ when $a = b$, and $x < y$ when $a < b$.

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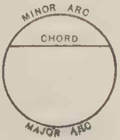
169. Chord. A straight line that has its extremities on a circle is called a *chord.*



A chord is said to *subtend* the arcs that it cuts from a circle. Unless the contrary is stated, the chord is taken as subtending the minor arc.

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PROPOSITION III. THEOREM

170. *In the same circle or in equal circles, if two arcs are equal, they are subtended by equal chords; and if two arcs are unequal, the greater is subtended by the greater chord.*

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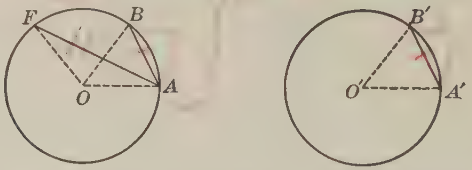


Given two equal circles with centers O and O', with arcs AB and $A'B'$ equal, and with arc AF greater than arc $A'B'$.

To prove that 1. *chord* $AB =$ *chord* $A'B'$;

2. *chord* $AF >$ *chord* $A'B'$.

Proof. 1. Draw the radii OA, OB, OF, $O'A'$, $O'B'$.

Since $OA = O'A'$, and $OB = O'B'$, § 162

and $\angle AOB = \angle A'O'B'$, § 167

(*In equal ⓢ equal arcs subtend equal central* ⦣*.*)

$\therefore \triangle OAB$ is congruent to $\triangle O'A'B'$, § 68

and chord $AB =$ chord $A'B'$. § 67

Proof. 2. In the ⦻ OAF and $O'A'B'$,

$OA = O'A'$, and $OF = O'B'$, § 162

but $\angle AOF$ is greater than $\angle A'O'B'$. § 167

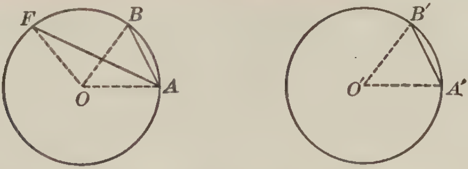
(*In equal ⓢ, of two unequal arcs the greater subtends the greater central* $\angle$.)

$\therefore$ chord $AF >$ chord $A'B'$, by § 115. Q.E.D.

171. COROLLARY. *In the same circle or in equal circles, the greater of two unequal major arcs is subtended by the less chord.*

PROPOSITION IV. THEOREM

172. *In the same circle or in equal circles, if two chords are equal, they subtend equal arcs; and if two chords are unequal, the greater subtends the greater arc.*



Given two equal circles with centers O and O', with chords AB and $A'B'$ equal, and with chord AF greater than chord $A'B'$.

To prove that 1. *arc* $AB =$ *arc* $A'B'$;

2. *arc* $AF >$ *arc* $A'B'$.

Proof. 1. Draw the radii OA, OB, OF, $O'A'$, $O'B'$.

Since $OA = O'A'$, and $OB = O'B'$, § 162

and chord $AB =$ chord $A'B'$, Given

$\therefore \triangle OAB$ is congruent to $\triangle O'A'B'$, § 80

and $\angle AOB = \angle A'O'B'$. § 67

$\therefore$ arc $AB =$ arc $A'B'$. § 166

Proof. 2. In the $\triangle$s OAF and $O'A'B'$,

$OA = O'A'$, and $OF = O'B'$, § 162

but chord $AF >$ chord $A'B'$. Given

$\therefore \angle AOF > \angle A'O'B'$. § 116

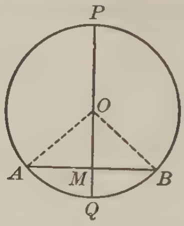
$\therefore$ arc $AF >$ arc $A'B'$, by § 166. Q.E.D.

This proposition is the converse of Prop. III.

173. COROLLARY. *In the same circle or in equal circles the greater of two unequal chords subtends the less major arc.*

PROPOSITION V. THEOREM

174. *A line through the center of a circle perpendicular to a chord bisects the chord and the arcs subtended by it.*



Given the line PQ through the center O of the circle $AQBP$, perpendicular to the chord AB at M.

To prove that $AM = BM$, *arc* $AQ =$ *arc* BQ, *and arc* $AP =$ *arc* BP.

Proof. Draw the radii OA and OB.

Then since $OM = OM$, Iden.

and $OA = OB$, § 162

$\therefore$ rt. $\triangle AMO$ is congruent to rt. $\triangle BMO$. § 89

$\therefore AM = BM$, and $\angle AOQ = \angle QOB$. § 67

Likewise $\angle POA = \angle BOP$. § 58

$\therefore$ arc $AQ =$ arc BQ, and arc $AP =$ arc BP, by § 166. Q.E.D.

175. COROLLARY 1. *A diameter bisects the circle.*

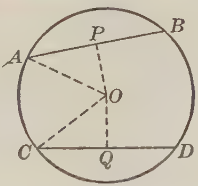
176. COROLLARY 2. *A line through the center that bisects a chord, not a diameter, is perpendicular to the chord.*

177. COROLLARY 3. *The perpendicular bisector of a chord passes through the center of the circle and bisects the arcs subtended by the chord.*

How many bisectors of the chord are possible? How many ⊥ bisectors? Therefore with what line must this coincide (§ 174)?

PROPOSITION VI. THEOREM

178. *In the same circle or in equal circles equal chords are equidistant from the center, and chords equidistant from the center are equal.*



Given AB and CD, equal chords of the circle $ACDB$.

To prove that AB and CD are equidistant from the center O.

Proof. Draw $OP \perp$ to AB, and $OQ \perp$ to CD.

Draw the radii OA and OC.

OP bisects AB, and OQ bisects CD. § 174

Then since $AP = CQ$, Ax. 4

and $OA = OC$, § 162

$\therefore$ rt. $\triangle OPA$ is congruent to rt. $\triangle OQC$ § 89

$\therefore OP = OQ$. § 67

$\therefore AB$ and CD are equidistant from O, by § 88. Q.E.D.

Given OP and OQ, equal perpendiculars from the center O to the chords AB and CD.

To prove that $AB = CD$.

Proof. Since $OA = OC$, § 162

and $OP = OQ$, Given

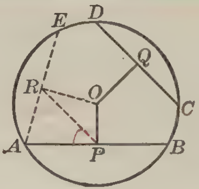
$\therefore$ rt. $\triangle OPA$ is congruent to rt. $\triangle OQC$. § 89

$\therefore AP = CQ$. § 67

$\therefore AB = CD$, by Ax. 3. Q.E.D.

PROPOSITION VII. THEOREM

179. *In the same circle or in equal circles, if two chords are unequal, they are unequally distant from the center, and the greater chord is at the less distance.*



Given a circle with center *O*, two unequal chords *AB* and *CD*, *AB* being the greater, and *OP* perpendicular to *AB*, and *OQ* perpendicular to *CD*.

To prove that $OP < OQ$.

Proof. Suppose AE drawn equal to CD, and $OR \perp$ to AE.

Draw PR.

OP bisects AB, and OR bisects AE. § 174

(*A line through the center of a circle $\perp$ to a chord bisects the chord.*)

But $AB > CD$. Given

$\therefore AB > AE$, the equal of CD. Ax. 9

$\therefore AP > AR$. Ax. 6

$\therefore \angle ARP > \angle RPA$. § 113

(*If two sides of a $\triangle$ are unequal, the $\angle$s opposite these sides are unequal, and the $\angle$ opposite the greater side is the greater.*)

$\therefore \angle PRO$, the complement of $\angle ARP$, is less than $\angle OPR$, the complement of $\angle RPA$. § 59

$\therefore OP < OR$. § 114

But $OR = OQ$. § 178

$\therefore OP < OQ$, by Ax. 9. Q.E.D.

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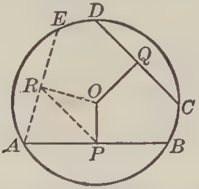
PROPOSITION VIII. THEOREM

180. *In the same circle or in equal circles, if two chords are unequally distant from the center, they are unequal, and the chord at the less distance is the greater.*



Given a circle with center O, two chords AB and CD unequally distant from O, and OP, the perpendicular to AB, less than OQ, the perpendicular to CD.

To prove that $AB > CD$.

Proof. Suppose AE drawn equal to CD, and $OR \perp$ to AE.

Now $OP < OQ$, Given

and $OR = OQ$. § 178

$\therefore OP < OR$. Ax. 9

Drawing PR, $\angle PRO < \angle OPR$. § 113

$\therefore \angle ARP$, the complement of $\angle PRO$, is greater than $\angle RPA$, the complement of $\angle OPR$. § 59

$\therefore AP > AR$. § 114

But $AP = \frac{1}{2} AB$, and $AR = \frac{1}{2} AE$. § 174

$\therefore AB > AE$. Ax. 6

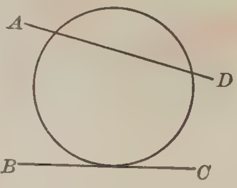
But $CD = AE$. Hyp.

$\therefore AB > CD$, by Ax. 9. Q.E.D.

This proposition is the converse of Prop. VII.

181. COROLLARY. *A diameter of a circle is greater than any other chord.*

182. Secant. A straight line that intersects a circle is called a *secant*. In this figure AD is a secant.



Since only two equal obliques can be drawn to a line from an external point (§ 85), and since the two equal angles which radii make (§ 74) with any secant where it cuts the circle cannot be right angles (§ 109), they must be oblique; and hence it follows that *a secant can intersect the circle in only two points.*

183. Tangent. A straight line of unlimited length that has one point, and only one, in common with a circle is called a *tangent* to the circle.

In this case the circle is said to be tangent to the line. Thus in the figure, BC is tangent to the circle, and the circle is tangent to BC.

The common point is called the *point of contact* or *point of tangency.*

By the tangent from an external point to a circle is meant the line-segment from the external point to the point of contact.

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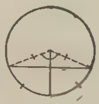
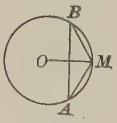
EXERCISE 27

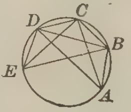


1. A radius that bisects an arc bisects its subtending chord and is perpendicular to it.

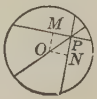
2. On a circle the point P is equidistant from two radii OA and OB. Prove that P bisects the arc AB.



3. In this circle the chords AM and MB are equal. Prove that M bisects the arc AB and that the radius OM bisects the chord AB.



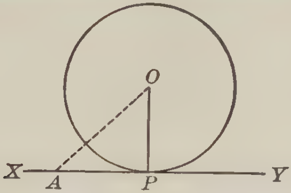
4. On a circle are five points, A, B, C, D, E, so placed that AB, BC, CD, DE are equal chords. Prove that AC, BD, CE are equal chords, and that AD and BE are also equal chords.



5. If two chords intersect and make equal angles with the diameter through their point of intersection, these chords are equal.

PROPOSITION IX. THEOREM

184. *A line perpendicular to a radius at its extremity on the circle is tangent to the circle.*



Given a circle, with XY perpendicular to the radius OP at P.

To prove that XY is tangent to the circle.

Proof. From O draw any other line to XY, as OA.

Then $OA > OP$. § 86

∴ the point A is outside the circle. § 160

Hence every point, except P, of the line XY is outside the circle.

Therefore XY is tangent to the circle at P, by § 183. Q.E.D.

185. COROLLARY 1. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

For OP is the shortest line from O to XY, and is therefore ⊥ to XY; that is, XY is ⊥ to OP.

186. COROLLARY 2. *A perpendicular to a tangent at the point of contact passes through the center of the circle.*

For a radius is ⊥ to a tangent at the point of contact, and therefore a ⊥ erected at the point of contact coincides with this radius and passes through the center of the circle.

187. COROLLARY 3. *A perpendicular from the center of a circle to a tangent passes through the point of contact.*

What does § 86 say about this perpendicular?

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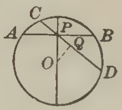
188. Concentric Circles. Two circles that have the same center are said to be *concentric.*

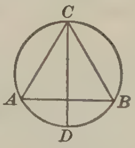
EXERCISE 28

1. The shortest chord that can be drawn through a given point within a circle is that which is perpendicular to the diameter through the point.



Show that any other chord, CD, through P, is nearer O than is AB.

2. The diameter CD bisects the arc AB. Prove that $\angle CBA = \angle BAC$.



What kind of a triangle is $\triangle ABC$?

3. Tangents at the extremities of a diameter are parallel.

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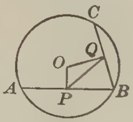
4. The arc AB is greater than the arc BC. OP and OQ are perpendiculars from the center to AB and BC respectively. Prove that $\angle QPO$ is greater than $\angle OQP$.

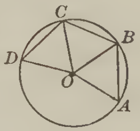


5. What is the locus of the center of a circle tangent to the line XY at the point P? Prove it.

What two conditions must be shown to be fulfilled?

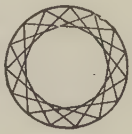
6. What is the locus of the mid-points of a number of parallel chords of a circle? Prove it.

7. Three equal chords, AB, BC, CD, are placed end to end, and the radii OA, OB, OC, OD are drawn. Prove that $\angle AOC = \angle BOD$.



8. All equal chords of a circle are tangent to a concentric circle.

9. If a number of equal chords are drawn in this circle, the figure gives the impression of a second circle inside the first and concentric with it. Explain the reason.



PROPOSITION X. THEOREM

189. *Two parallel lines intercept equal arcs on a circle.*

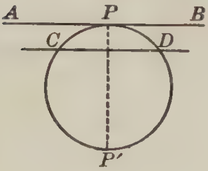


FIG. 1

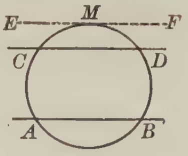


FIG. 2

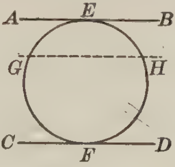


FIG. 3

CASE 1. *When the parallels are a tangent and a secant* (Fig. 1).

Given AB, a tangent at P, parallel to CD, a secant.

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To prove that arc CP = arc DP.

Proof. Suppose PP' drawn ⊥ to AB at P.

Then PP' is a diameter of the circle. § 186

And PP' is also ⊥ to CD. § 97

∴ arc CP = arc DP. § 174

CASE 2. *When the parallels are both secants* (Fig. 2).

Given AB and CD, parallel secants.

To prove that arc AC = arc BD.

Proof. Suppose EF ∥ to CD and tangent to the circle at M.

Then arc AM = arc BM, and arc CM = arc DM. Case 1

∴ arc AC = arc BD. Ax. 2

CASE 3. *When the parallels are both tangents* (Fig. 3).

Given AB, a tangent at E, parallel to CD, a tangent at F.

To prove that arc FGE = arc FHE.

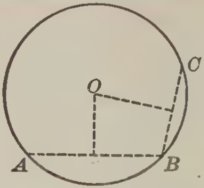
Proof. Suppose a secant GH drawn ∥ to AB.

Then arc GE = arc HE, and arc FG = arc FH. Case 1

∴ arc FGE = arc FHE, by Ax. 1. Q. E. D.

PROPOSITION XI. THEOREM

190. *Through three points not in a straight line one circle, and only one, can be drawn.*



Given A, B, C, three points not in a straight line.

To prove that one circle, and only one, can be drawn through A, B, and C.

Proof. Draw AB and BC.

At the mid-points of AB and BC suppose ⊥s erected.

These ⊥s will intersect at some point O, since AB and BC are neither parallel nor in the same straight line.

The point O is in the perpendicular bisector of AB, and is therefore equidistant from A and B; the point O is also in the perpendicular bisector of BC, and is therefore equidistant from B and C. § 150

Therefore O is equidistant from A, B, and C.

Therefore a circle described about O as a center, with a radius OA, will pass through the three given points. § 160

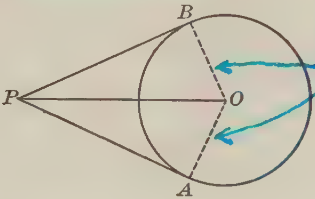
The center of any circle that passes through the three points must be in both of these perpendicular bisectors, and hence at their intersection. As two straight lines can intersect in only one point (§ 55), O is the only point that can be the center of a circle through the three given points. Q. E. D.

191. COROLLARY. *Two circles can intersect in only two points.*

If two circles have three points in common, can it be shown that they coincide and form one circle?

PROPOSITION XII. THEOREM

192. *The tangents to a circle drawn from an external point are equal, and make equal angles with the line joining the point to the center.*



radii of same $\odot$ are =

Given PA **and** PB**, tangents from** P **to the circle whose center is** O**, and** PO **the line joining** P **to the center** O**.**

To prove that $PA = PB$, *and* $\angle APO = \angle OPB$.

Proof. Draw OA and OB.

PA is $\perp$ to OA, and PB is $\perp$ to OB. § 185

(*A tangent to a circle is* $\perp$ *to the radius drawn to the point of contact.*)

In the rt. $\triangle$ PAO and PBO,

$PO = PO$, Iden.

and $OA = OB$. § 162

$\therefore$ rt. $\triangle PAO$ is congruent to rt. $\triangle PBO$. § 89

$\therefore$ $PA = PB$, and $\angle APO = \angle OPB$, by § 67. Q.E.D.

193. Line of Centers. The line determined by the centers of two circles is called the *line of centers.*

194. Tangent Circles. Two circles that are both tangent to the same line at the same point are called *tangent circles.*

Circles are said to be tangent *internally* or *externally*, according as they lie on the same side of the tangent line or on opposite sides. E.g. the two circles shown in the figure on page 110 are tangent externally.

The point of contact with the line is called the *point of contact* or *point of tangency* of the circles.

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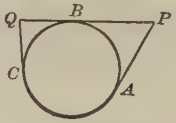
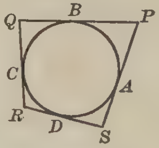
EXERCISE 29

1. Show that the reasoning of § 190 will not hold for four points, and hence that a circle cannot always be drawn through four points.

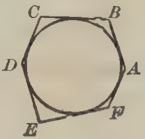


2. Tangents to a circle at A, B, C, points on the circle, meet in P and Q, as here shown. Prove that $AP + QC = PQ$.

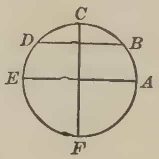


3. If a quadrilateral has each side tangent to a circle, the sum of one pair of opposite sides equals the sum of the other pair.

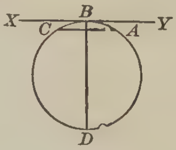
In this figure, $SP + QR = PQ + RS$.



4. The hexagon here shown has each side tangent to the circle. Prove that $AB + CD + EF = BC + DE + FA$.

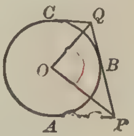


5. In this figure CF is a diameter perpendicular to the parallel chords DB and EA, and arc $AB = 40°$ and arc $BC = 50°$. How many degrees are there in arcs CD, DE, EF, and FA?



6. In this figure XY is tangent to the circle at B, the chord CA is perpendicular to the diameter BD, and the arc $CD = 150°$. How many degrees are there in arc AB?

7. If a quadrilateral has each side tangent to a circle, the sum of the angles at the center subtended by any two opposite sides is equal to a straight angle.

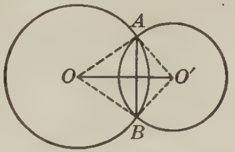


8. AP and CQ are parallel tangents meeting a third tangent QP, as shown in the figure. O being the center, prove that the angle POQ is a right angle.

Are A, O, and C in the same straight line? Draw OA and OC, and find the relations of the ∠s at O to those at P and Q.

PROPOSITION XIII. THEOREM

195. *If two circles intersect, the line of centers is the perpendicular bisector of their common chord.*



Given O and O', the centers of two intersecting circles, AB the common chord, and OO' the line of centers.

To prove that OO' is $\perp$ to AB at its mid-point.

Proof. Draw OA, OB, $O'A$, and $O'B$.

$OA = OB$, and $O'A = O'B$. § 162

$\therefore$ O and O' are two points, each equidistant from A and B.

$\therefore$ OO' is the perpendicular bisector of AB, by § 151. Q.E.D.

196. Common Tangents. A tangent to two circles is called a *common external tangent* if it does not cut the line-segment joining the centers, and a *common internal tangent* if it cuts it.

EXERCISE 30

Describe the relative position of two circles if the line-segment joining the centers is related to the radii as stated in Exs. 1–5, and illustrate each case by a figure:

1. The line-segment greater than the sum of the radii.

2. The line-segment equal to the sum of the radii.

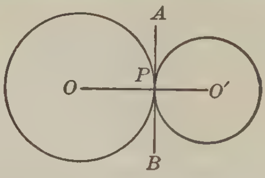
3. The line-segment less than the sum but greater than the difference of the radii.

4. The line-segment equal to the difference of the radii.

5. The line-segment less than the difference of the radii.

PROPOSITION XIV. THEOREM

197. *If two circles are tangent to each other, the line of centers passes through the point of contact.*



Given two circles tangent at P.

To prove that P is in the line of centers.

Proof. Let AB be the common tangent at P. § 194

Then a ⊥ to AB, drawn through the point P, passes through the centers O and O'. § 186

(*A ⊥ to a tangent at the point of contact passes through the center of the circle.*)

Therefore the line determined by O and O', having two points in common with this ⊥, must coincide with it. Post. 1

∴ P is in the line of centers. Q.E.D.

EXERCISE 31

Describe the relative position of two circles having tangents as stated in Exs. 1–5, and illustrate each case by a figure:

1. Two common external and two common internal tangents.
2. Two common external tangents and one common internal tangent.
3. Two common external tangents and no common internal tangent.
4. One common external and no common internal tangent.
5. No common tangent.

6. The line which passes through the mid-points of two parallel chords passes through the center of the circle.

7. If two circles are tangent externally, the tangents to them from any point of the common internal tangent are equal.

8. If two circles tangent externally are tangent to a line AB at A and B, their common internal tangent bisects AB.

9. The line drawn from the center of a circle to the point of intersection of two tangents is the perpendicular bisector of the chord joining the points of contact.

10. The diameters of two circles are respectively 2.74 in. and 3.48 in. Find the distance between the centers of the circles if they are tangent externally. Find the distance between the centers of the circles if they are tangent internally.

11. Three circles of diameters 4.8 in., 3.6 in., and 4.2 in. are externally tangent, each to the other two. Find the perimeter of the triangle formed by joining the centers.

12. A circle of center O and radius r' rolls around a fixed circle of radius r. What is the locus of O? Prove it.

13. The line drawn from the mid-point of a chord to the mid-point of its subtended arc is perpendicular to the chord.

14. If two circles tangent externally at P are tangent to a line AB at A and B, the angle BPA is a right angle.

15. Three circles are tangent externally at the points A, B, and C, and the chords AB and AC are produced to cut the circle BC at D and E. Prove that DE is a diameter.

16. If two radii of a circle, at right angles to each other, when produced are cut by a tangent to the circle at A and B, the other tangents from A and B are parallel to each other.

17. If two common external tangents or two common internal tangents are drawn to two circles, the line-segments intercepted between the points of contact are equal.

198. Measure. The number of times a quantity of any kind contains a known unit of the same kind, expressed in terms of that known unit, is called the *measure* of the quantity.

Thus we measure the *length* of a schoolroom by finding the number of times it contains a *known unit* called the *foot.* We measure the *area* of the floor by finding the number of times it contains a *known unit* called the *square foot.* You measure your weight by finding the number of times it contains a *known unit* called the *pound.* Thus the measure of the length of a room may be 30 ft., the measure of the area of the floor may be 600 sq. ft., and so on.

The abstract number found in measuring a quantity is called its *numerical measure,* or usually simply its *measure.*

199. Ratio. The quotient of the numerical measures of two quantities, expressed in terms of a common unit, is called the *ratio* of the quantities.

Thus, if a room is 20 ft. by 35 ft., the ratio of the width to the length is 20 ft. ÷ 35 ft., or $\frac{20}{35}$, which reduces to $\frac{4}{7}$. Here the *common unit* is 1 ft.

The ratio of a to b is written $\frac{a}{b}$, or $a : b$, as in arithmetic and algebra.

Thus the ratio of 20° to 30° is $\frac{20}{30}$, or $\frac{2}{3}$, or 2 : 3.

200. Commensurable Magnitudes. Two quantities of the same kind that can both be expressed in integers in terms of a common unit are said to be *commensurable magnitudes.*

Thus 20 ft. and 35 ft. are expressed in integers (20 and 35) in terms of a common unit (1 ft.); similarly 2 ft. and $3\frac{1}{2}$ ft., the integers being 4 and 7, and the common unit being $\frac{1}{2}$ ft.

The common unit used in measuring two or more commensurable magnitudes is called their *common measure.* Each of the magnitudes is called a *multiple* of this common measure.

201. Incommensurable Magnitudes. Two quantities of the same kind that cannot both be expressed in integers in terms of a common unit are said to be *incommensurable magnitudes.*

Thus, if $a = \sqrt{2}$ and $b = 3$, there is no number that is contained an integral number of times in both $\sqrt{2}$ and 3. Hence a and b are, in this case, incommensurable magnitudes.

202. Incommensurable Ratio. The ratio of two incommensurable magnitudes is called an *incommensurable ratio.*

Although the exact value of such a ratio cannot be expressed by an integer, a common fraction, or a decimal fraction of a limited number of places, it may be expressed *approximately.*

Thus suppose $\frac{a}{b} = \sqrt{2}$.

Now $\sqrt{2} = 1.41421356 \cdots$, which is greater than 1.414213 but less than 1.414214. Then if a *millionth part* of b is taken as the unit of measure, the value of $a : b$ lies between 1.414213 and 1.414214, and therefore differs from either by less than 0.000001.

By carrying the decimal further an approximate value may be found that will differ from the ratio by less than a *billionth,* a *trillionth,* or *any other assigned value.*

That is, *for practical purposes all ratios are commensurable.*

For example, if $\frac{a}{b} > \frac{m}{n}$ but $< \frac{m+1}{n}$, then the error in taking either of these values for $\frac{a}{b}$ is less than $\frac{1}{n}$, the difference of these ratios. But by increasing n indefinitely, $\frac{1}{n}$ can be decreased indefinitely, and a value of the ratio can be found within any required degree of accuracy.

EXERCISE 32

Find a common measure of:

1. 32 in., 24 in. **3.** $5\frac{1}{2}$ in., $3\frac{1}{2}$ in. **5.** $6\frac{1}{3}$ da., $2\frac{2}{3}$ da.

2. 48 ft., 18 ft. **4.** $2\frac{3}{4}$ lb., $1\frac{1}{2}$ lb. **6.** 14.4 in., 1.2 in.

Find the greatest common measure of:

7. 64 yd., 24 yd. **9.** 7.5 in., 1.25 in. **11.** $2\frac{3}{4}$ ft., 0.25 ft.

8. 51 ft., 17 ft. **10.** $3\frac{1}{3}$ in., $0.33\frac{1}{3}$ in. **12.** 75°, 7° 30′.

13. If $a : b = \sqrt{3}$, find an approximate value of this ratio that shall differ from the true value by less than 0.001.

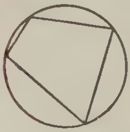
203. Constant and Variable. A quantity regarded as having a fixed value throughout a given discussion is called a *constant*, but a quantity regarded as having different successive values is called a *variable*.

204. Limit. When a variable approaches a constant in such a way that the difference between the two may become and remain less than any assigned positive quantity, however small, the constant is called the *limit* of the variable.

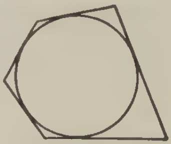
Variables can sometimes reach their limits and sometimes not. E.g. a chord may increase in length up to a certain limit, the diameter, and it can reach this limit and still be a chord; it may decrease, approaching the limit 0, but it cannot reach this limit and still be a chord.

205. Inscribed and Circumscribed Polygons. If the sides of a polygon are all chords of a circle, the polygon is said to be *inscribed* in the circle; if the sides are all tangents to a circle, the polygon is said to be *circumscribed* about the circle.

The circle is said to be *circumscribed* about the inscribed polygon, and to be *inscribed* in the circumscribed polygon.

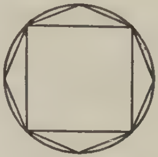


Inscribed Polygon



Circumscribed Polygon

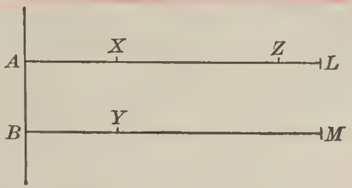
206. Circle as a Limit. If we inscribe a square in a circle, and then inscribe an octagon by taking the mid-points of the four equal arcs for the new vertices, the octagon is greater than the square but smaller than the area inclosed by the circle, and the perimeter of the octagon is greater than the perimeter of the square (§ 112).



By continually doubling the number of sides in this way it appears that the area inclosed by the circle is the limit of the area of the polygon, and the circle is the limit of its perimeter, as the number of sides is indefinitely increased.

Hence we have limiting *forms* as well as limiting *values*, the form of the circle being the limit approached by the form of the inscribed polygon.

207. Principle of Limits. *If, while approaching their respective limits, two variables are always equal, their limits are equal.*



Let AX and BY increase in length in such a way that they always remain equal, and let their respective limits be AL and BM.

To prove that $AL = BM$.

Suppose these limits are not equal, but that $AZ = BM$.

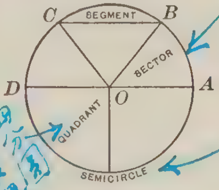
Then since X may reach a point between Z and L we may have $AX > AZ$, and therefore greater than its supposed equal, BM; but BY cannot be greater than BM. Therefore we should have $AX > BY$, which is contrary to what is given.

Hence AL cannot be greater than BM, and similarly BM cannot be greater than AL. $\therefore AL = BM$. Q. E. D.

208. Area of Circle. The area inclosed by a circle is called the *area of the circle.*

It is evident that a *diameter bisects the area of a circle.*

209. Segment. A portion of a plane bounded by an arc of a circle and its chord is called a *segment* of the circle.



If the chord is a diameter, the segment is called a *semicircle*, this word being commonly used to mean not only half of the circle but also the area inclosed by a semicircle and a diameter.

210. Sector. A portion of a plane bounded by two radii and the arc of the circle intercepted by the radii is called a *sector*.

If the arc is a quarter of the circle, the sector is called a *quadrant*.

211. Inscribed Angle. An angle whose vertex is on a circle, and whose sides are chords, is called an *inscribed angle*.

An angle is said to be *inscribed in a segment* if its vertex is on the arc of the segment and its sides pass through the ends of the arc.

PROPOSITION XV. THEOREM

212. *In the same circle or in equal circles two central angles have the same ratio as their intercepted arcs.*

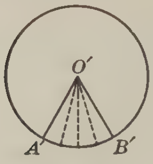


FIG. 1

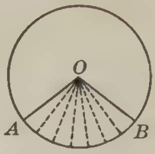


FIG. 2

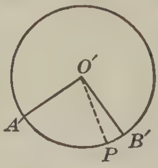


FIG. 3

Given two equal circles with centers O and O', AOB and $A'O'B'$ being central angles, and AB and $A'B'$ the intercepted arcs.

To prove that $\dfrac{\angle A'O'B'}{\angle AOB} = \dfrac{\text{arc } A'B'}{\text{arc } AB}$.

CASE 1. *When the arcs are commensurable* (Figs. 1 and 2).

Proof. Let the arc m be a common measure of $A'B'$ and AB.

Apply the arc m as a measure to the arcs $A'B'$ and AB as many times as they will contain it.

Suppose m is contained a times in $A'B'$, and b times in AB.

Then $\dfrac{\text{arc } A'B'}{\text{arc } AB} = \dfrac{a}{b}$.

At the several points of division on AB and $A'B'$ draw radii.

These radii will divide $\angle AOB$ into b parts, and $\angle A'O'B'$ into a parts, equal each to each. § 167

$$\therefore \frac{\angle A'O'B'}{\angle AOB} = \frac{a}{b}.$$

$$\therefore \frac{\angle A'O'B'}{\angle AOB} = \frac{\text{arc } A'B'}{\text{arc } AB}, \text{ by Ax. 8.}$$ Q.E.D.

Case 2 may be omitted at the discretion of the teacher if the incommensurable cases are not to be taken in the course.

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CASE 2. *When the arcs are incommensurable* (Figs. 2 and 3).

Proof. Divide AB into a number of equal parts; and apply one of these parts to $A'B'$ as many times as $A'B'$ will contain it.

Since AB and $A'B'$ are incommensurable, a certain number of these parts will extend from A' to some point, as P, leaving a remainder PB' less than one of these parts. Draw $O'P$.

By construction AB and $A'P$ are commensurable.

$$\therefore \frac{\angle A'O'P}{\angle AOB} = \frac{\text{arc } A'P}{\text{arc } AB}. \qquad \text{Case 1}$$

By increasing the *number* of equal parts into which AB is divided we can diminish the *length* of each, and therefore can make PB' less than any assigned positive value, however small.

Hence PB' approaches zero as a limit as the number of parts of AB is indefinitely increased, and at the same time the corresponding angle $PO'B'$ approaches zero as a limit. § 204

Therefore the arc $A'P$ approaches the arc $A'B'$ as a limit, and the $\angle A'O'P$ approaches the $\angle A'O'B'$ as a limit.

$\therefore$ the variable $\dfrac{\text{arc } A'P}{\text{arc } AB}$ approaches $\dfrac{\text{arc } A'B'}{\text{arc } AB}$ as a limit,

and the variable $\dfrac{\angle A'O'P}{\angle AOB}$ approaches $\dfrac{\angle A'O'B'}{\angle AOB}$ as a limit.

But $\dfrac{\angle A'O'P}{\angle AOB}$ is always equal to $\dfrac{\text{arc } A'P}{\text{arc } AB}$,

as $A'P$ varies in value and approaches $A'B'$ as a limit. Case 1

$$\therefore \frac{\angle A'O'B'}{\angle AOB} = \frac{\text{arc } A'B'}{\text{arc } AB}, \text{ by § 207.} \qquad \text{Q.E.D.}$$

213. Numerical Measure. We therefore see that the numerical measure of a central angle (in degrees, for example) equals the numerical measure of the intercepted arc. This is commonly expressed by saying that *a central angle is measured by the intercepted arc.*

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内容 内接

PROPOSITION XVI. THEOREM

214. *An inscribed angle is measured by half the intercepted arc.*

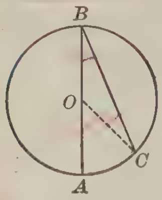


FIG. 1

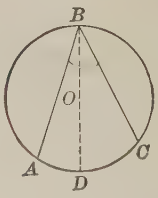


FIG. 2

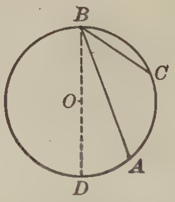


FIG. 3

Given a circle with center *O* and the inscribed angle *B*, intercepting the arc *AC*.

To prove that $\angle B$ is measured by half the arc AC.

CASE 1. *When O is on one side, as AB* (Fig. 1).

Proof. Draw OC.

Then $\because OC = OB$, § 162

$\therefore \angle B = \angle C$. § 74

But $\angle B + \angle C = \angle AOC$. § 111

$\therefore 2\angle B = \angle AOC$. Ax. 9

$\therefore \angle B = \frac{1}{2}\angle AOC$. Ax. 4

But $\angle AOC$ is measured by arc AC. § 213

$\therefore \frac{1}{2}\angle AOC$ is measured by $\frac{1}{2}$ arc AC. Ax. 4

$\therefore \angle B$ is measured by $\frac{1}{2}$ arc AC. Ax. 9

CASE 2. *When O lies within the angle B* (Fig. 2).

Proof. Draw the diameter BD.

Then $\angle ABD$ is measured by $\frac{1}{2}$ arc AD,

and $\angle DBC$ is measured by $\frac{1}{2}$ arc DC. Case 1

$\therefore \angle ABD + \angle DBC$ is measured by $\frac{1}{2}$ (arc AD + arc DC),

or $\angle ABC$ is measured by $\frac{1}{2}$ arc AC.

CASE 3. *When O lies outside the angle B* (Fig. 3)

Proof. Draw the diameter BD.

Then $\angle DBC$ is measured by $\frac{1}{2}$ arc DC,

and $\angle DBA$ is measured by $\frac{1}{2}$ arc DA. Case 1

$\therefore \angle DBC - \angle DBA$ is measured by $\frac{1}{2}$ (arc DC – arc DA),

or $\angle ABC$ is measured by $\frac{1}{2}$ arc AC. Q.E.D.

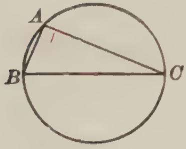


FIG. 4

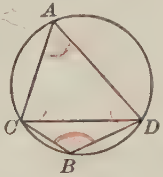


FIG. 5

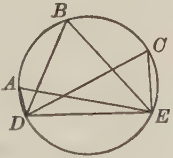


FIG. 6

215. COROLLARY 1. *An angle inscribed in a semicircle is a right angle.*

For it is half of a central straight angle, as in Fig. 4.

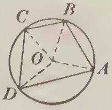
216. COROLLARY 2. *An angle inscribed in a segment greater than a semicircle is an acute angle, and an angle inscribed in a segment less than a semicircle is an obtuse angle.*

See ∡ A and B in Fig. 5.

217. COROLLARY 3. *Angles inscribed in the same segment or in equal segments are equal.*

Why is this? (Fig. 6.)

218. COROLLARY 4. *If a quadrilateral is inscribed in a circle, the opposite angles are supplementary; and, conversely, if two opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.*



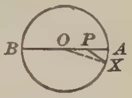
For the second part, can a circle be passed through A, B, C (§ 190)? If it does not pass through D also, can you show that $\angle D$ would be greater than or less than some other angle (§ 111) that is supplementary to $\angle B$?

EXERCISE 33

1. A parallelogram inscribed in a circle is a rectangle.

2. A trapezoid inscribed in a circle is isosceles.

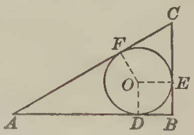
3. The shorter segment of the diameter through a given point within a circle is the shortest line that can be drawn from that point to the circle.



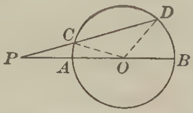
Let P be the given point. Prove PA shorter than any other line PX from P to the circle.

4. The longer segment of the diameter through a given point within a circle is the longest line that can be drawn from that point to the circle.

5. The diameter of the circle inscribed in a right triangle is equal to the difference between the hypotenuse and the sum of the other two sides.



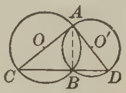
6. A line from a given point outside a circle passing through the center contains the shortest line-segment that can be drawn from that point to the circle.



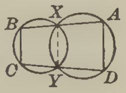
Let P be the point, O the center, A the point where PO cuts the circle, and C any other point on the circle. How does $PC+CO$ compare with PO?

7. A line from a given point outside a circle passing through the center contains the longest line-segment (to the concave arc) that can be drawn from that point to the circle.

8. Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the line joining the ends of the diameters passes through the other point of intersection.

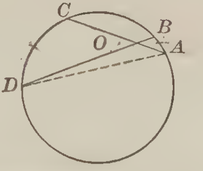


9. If two circles intersect and a line is drawn through each point of intersection terminated by the circles, the chords joining the corresponding ends of these lines are parallel.



PROPOSITION XVII. THEOREM

219. *An angle formed by two chords intersecting within the circle is measured by half the sum of the intercepted arcs.*



Given the angle *AOB* formed by the chords *AC* and *BD*.

To prove that $\angle AOB$ *is measured by* $\frac{1}{2}$ *(arc AB + arc CD).*

Proof. Draw *AD*.

Then $\angle AOB = \angle A + \angle D$. § 111

(*An exterior ∠ of a △ is equal to the sum of the two opposite interior ∠s.*)

But $\angle A$ is measured by $\frac{1}{2}$ arc *CD*, § 214

(*An inscribed ∠ is measured by half the intercepted arc.*)

and $\angle D$ is measured by $\frac{1}{2}$ arc *AB*. § 214

∴ $\angle AOB$ is measured by $\frac{1}{2}$(arc *AB* + arc *CD*), by Ax. 1. Q.E.D.

Discussion. If *O* is at the center of the circle, to what previous proposition does this proposition reduce?

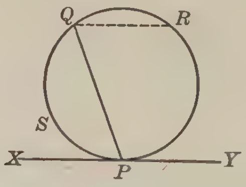
If *O* is on the circle, as at *B*, to what previous proposition does this proposition reduce?

Suppose the point *O* remains as in the figure, and the chord *AC* swings about *O* as a pivot until it coincides with the chord *BD*. What can then be said of the measure of ∠s *AOB* and *COD*? What can be said as to the measure of ∠s *BOC* and *DOA*?

It is also possible to prove the proposition by drawing a chord *AE* parallel to *BD*, and showing that $\angle AOB = \angle A$, since they are alternate-interior angles formed by a transversal cutting two parallels. Now $\angle A$ is measured by $\frac{1}{2}$ arc *CE*. But arc *CE* = arc *CD* + arc *DE*, or arc *CD* + arc *AB*, since arc *AB* = arc *DE* (§ 189). Therefore $\angle AOB$, which equals $\angle A$, is measured by $\frac{1}{2}$(arc *AB* + arc *CD*).

PROPOSITION XVIII. THEOREM

220. *An angle formed by a tangent and a chord drawn from the point of contact is measured by half the intercepted arc.*



Given the chord PQ and the tangent XY through P.

To prove that $\angle QPX$ *is measured by half the arc* QSP.

Proof. Suppose the chord QR drawn from the point Q parallel to the tangent XY.

Then arc $PR =$ arc QSP. § 189

(*Two parallel lines intercept equal arcs on a circle.*)

Also $\angle QPX = \angle PQR$. § 100

(*If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*)

But $\angle PQR$ is measured by $\frac{1}{2}$ arc PR. § 214

(*An inscribed $\angle$ is measured by half the intercepted arc.*)

Substitute $\angle QPX$ for its equal, the $\angle PQR$,

and substitute arc QSP for its equal, the arc PR.

Then $\angle QPX$ is measured by $\frac{1}{2}$ arc QSP, by Ax. 9. Q.E.D.

Discussion. By half of what arc is $\angle YPQ$, the supplement of $\angle QPX$, measured?

If PQ should be drawn so as to be perpendicular to XY, by what would $\angle s$ YPQ and QPX be measured?

Suppose PQ swings about the point P as a pivot until it coincides with XY, by what will $\angle YPQ$ be measured? By what will $\angle QPX$ be measured, and what will it equal?

PROPOSITION XIX. THEOREM

221. *An angle formed by two secants, a secant and a tangent, or two tangents, drawn to a circle from an external point, is measured by half the difference of the intercepted arcs.*

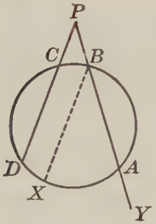


FIG. 1

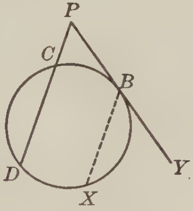


FIG. 2

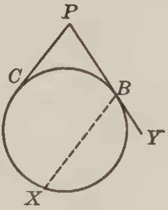


FIG. 3

Given two secants PBA and PCD, from the external point P.

To prove that $\angle P$ *is measured by* $\frac{1}{2}$ *(arc* DA *– arc* BC*).*

Proof. Suppose the chord BX drawn ∥ to PCD (Fig. 1).

Then arc $BC =$ arc DX. § 189

Furthermore arc $XA =$ arc $DA -$ arc DX.

$\therefore$ arc $XA =$ arc $DA -$ arc BC. Ax. 9

Also $\angle P = \angle XBA$. § 102

But $\angle XBA$ is measured by $\frac{1}{2}$ arc XA. § 214

Substitute $\angle P$ for its equal, the $\angle XBA$,

and substitute arc $DA -$ arc BC for its equal, the arc XA.

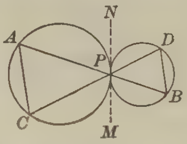
Then $\angle P$ is measured by $\frac{1}{2}$ (arc $DA -$ arc BC), by Ax. 9. Q.E.D.

If the secant $PBAY$ swings around to tangency, it becomes the tangent PB and Fig. 1 becomes Fig. 2. If the secant PCD also swings around to tangency, it becomes the tangent PC and Fig. 2 becomes Fig. 3. The proof of the theorem for each of these cases is left for the student.

EXERCISE 34

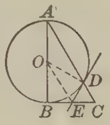
1. If two circles touch each other and two lines are drawn through the point of contact terminated by the circles, the chords joining the ends of these lines are parallel.

This could be proved if it could be shown that $\angle A$ equals what angle? To what two angles can these angles be proved equal by § 220? Are those angles equal?

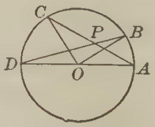


2. If one side of a right triangle is the diameter of a circle, the tangent at the point where the circle cuts the hypotenuse bisects the other side.

If OE is $\parallel$ to AC, then because $BO = OA$, what is the relation of BE to EC? The proposition therefore reduces to proving that OE is parallel to what line? This can be proved if $\angle BOE$ can be shown equal to what angle?

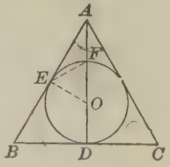


3. If from the extremities of a diameter AD two chords, AC and DB, are drawn intersecting at P so as to make $\angle APB = 45°$, then $\angle BOC$ is a right angle.



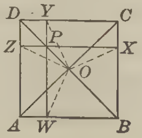
4. The radius of the circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.

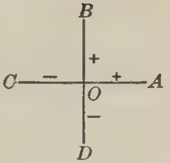
To prove this we must show that AF equals what line? It looks as if AF might equal EF, and EF equal OF. Is there any way of proving $\triangle OFE$ equilateral? of proving $\triangle AEF$ isosceles?



5. If two lines are drawn through any point in a diagonal of a square parallel to the sides, the points where these lines meet the sides lie on the circle whose center is the point of intersection of the diagonals.

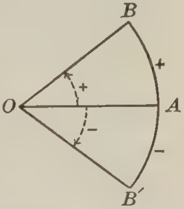
$OY = OZ$ if what two $\triangle$s are congruent? Why are these $\triangle$s congruent? $OY = OX$ if what two $\triangle$s are congruent? $OX = OW$ if what two $\triangle$s are congruent?





222. Positive and Negative Quantities. In geometry, as in algebra, quantities may be distinguished as *positive* and as *negative.*

Thus as we consider temperature above zero positive and temperature below zero negative, so in this figure, if OB is considered *positive*, then OD may be considered *negative.* Similarly, if OA is considered positive, then OC may be considered negative.

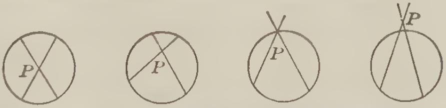


Likewise with respect to angles and arcs, if the rotating line OA moves in the direction of AB, counterclockwise, the angle and arc generated are considered *positive.* If it rotates in the direction AB', like the hands of a clock, the angle and arc generated are considered *negative.*

223. Principle of Continuity. By considering the distinction between positive and negative magnitudes, a theorem may often be so stated as to include several particular theorems. For example, *The angle included between two lines that cut or are tangent to a circle is measured by half the sum of the intercepted arcs.*

In particular: 1. If the lines intersect at the center, half the sum of the arcs will then become simply one of the arcs, and the proposition reduces to that of § 213.

2. If the lines are two general chords, we have the case of § 219.



3. If the point of intersection P moves to the circle, we have the case of § 214, one arc becoming zero.

4. If P moves outside the circle, then the smaller arc passes through zero and becomes negative, so that the sum of the arcs becomes their arithmetical difference (§ 221).

We may continue the discussion so as to include all the cases of the propositions proved from § 213 to § 221.

When the reasoning employed to prove a theorem is continued as just illustrated, so as to include several theorems, we are said to reason by the *Principle of Continuity.*

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224. Problems of Construction. At the beginning of the study of geometry some directions were given for simple constructions, so that figures might be drawn with accuracy. It was not proved at that time that these constructions were correct, because no theorems had been studied on which proofs could be based. It is now purposed to review these constructions, to prove that they are correct, and to apply the methods employed to the solution of more difficult problems.

225. Nature of a Solution. A solution of a problem has one requirement that a proof of a theorem does not have.

In a theorem we have three general steps: (1) *Given*, (2) *To prove*, (3) *Proof*. In a problem we have four steps: (1) *Given*, (2) *Required* (to do some definite thing), (3) *Construction* (showing how to do it), (4) *Proof* (that the construction is correct).

We *prove* a theorem, but we *solve* a problem, and then prove that our solution is a correct one.

In the figures of this text given lines are shown as full, black lines; construction lines and lines required are shown as dotted lines.

226. Discussion of a Problem. Besides the four necessary general steps in treating a problem, there is a desirable step to be taken in many cases. This is the discussion of the problem, in which is considered whether there is more than one solution, and other similar questions.

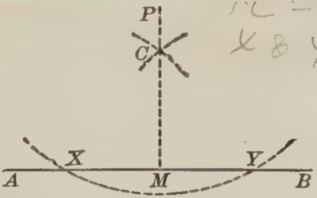
For example, suppose the problem is this: *Required from a given point to draw a tangent to a circle.*

After the problem has been solved we may discuss it thus: In general, if the given point is outside the circle, two tangents may be drawn, and these tangents are equal (§ 192); if the given point is on the circle only one tangent can be drawn, since only one perpendicular can be drawn to a radius at its extremity (§ 184); if the given point is within the circle, evidently no tangent can be drawn.

In the discussion the Principle of Continuity often enters, the figure being studied for various positions of some given point or line, as was done in the discussions on pages 121 and 122.

Proposition XX. Problem

227. *To let fall a perpendicular upon a given line from a given external point.*



PC = distant from X & Y

Given the line ***AB*** **and the external point** ***P*.**

Required from P to let fall a ⊥ upon AB.

Construction. With P as a center, and a radius sufficiently great, describe an arc cutting AB at X and Y. Post. 4

With X and Y as centers, and a convenient radius, describe two arcs intersecting at C. Post. 4

Draw PC. Post. 1

Produce PC to intersect AB at M. Post. 2

Then PM is the line required. Q.E.F.

Proof. Since P and C are by construction two points each equidistant from X and Y, they determine the perpendicular to XY at its mid-point. § 151

(*Two points each equidistant from the extremities of a line determine the ⊥ bisector of the line.*) Q.E.D.

Discussion. The following are interesting considerations:

That PC produced will really intersect AB, as stated in the construction, is shown in the proof.

A convenient radius to take for the two intersecting arcs is XY.

If C falls on P, take C at the other intersection of the arcs below AB, as is seen in the figure of Ex. 2, p. 9.

To obtain a radius for the first circle, draw any line from P that will cut AB, and use that.

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PROPOSITION XXI. PROBLEM

228. *At a given point in a given line, to erect a perpendicular to the line.*

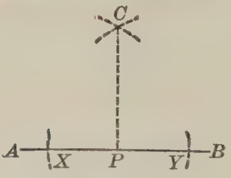


FIG. 1

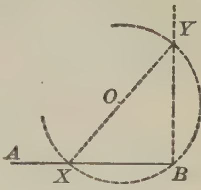


FIG. 2

Given the point P in the line AB.

Required to erect a ⊥ to AB at P.

CASE 1. *When the point P is not at the end of AB* (Fig. 1).

Construction. Take $PX = PY$. Post. 4

With X and Y as centers, and a convenient radius, describe arcs intersecting at C. Post. 4

Draw CP. Post. 1

Then CP is ⊥ to AB. Q.E.F.

Proof. P and C, two points each equidistant from X and Y, determine the ⊥ bisector of XY, by § 151. Q.E.D.

CASE 2. *When the point P is at the end of AB* (Fig. 2).

Construction. Suppose P to coincide with B.

Take any point O outside of AB, and with O as a center and OB as a radius describe a circle intersecting AB at X.

From X draw the diameter XY, and draw BY. Post. 1

Then BY is ⊥ to AB. Q.E.F.

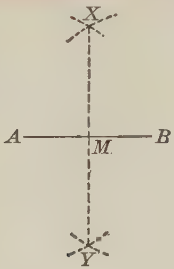
Proof. $\angle B$ is a right angle. § 215

∴ BY is ⊥ to AB, by § 27. Q.E.D.

Discussion. If the circle described with O as a center is tangent to AB at B, then OB is the required perpendicular (§ 185).

PROPOSITION XXII. PROBLEM

229. *To bisect a given line.*



Given the line AB.

Required to bisect AB.

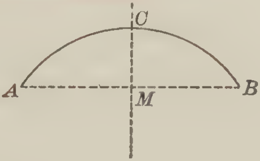
Construction. With A and B as centers and AB as a radius describe arcs intersecting at X and Y, and draw XY. Post. 4

Then XY bisects AB. Q. E. F.

Proof. XY bisects AB, by § 151. Q. E. D.

PROPOSITION XXIII. PROBLEM

230. *To bisect a given arc.*



Given the arc AB.

Required to bisect AB.

Construction. Draw the chord AB. Post. 1

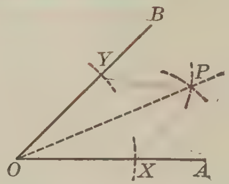
Draw CM, the perpendicular bisector of the chord AB. § 229

Then CM bisects the arc AB. Q. E. F.

Proof. CM bisects the arc AB, by § 177. Q. E. D.

PROPOSITION XXIV. PROBLEM

231. *To bisect a given angle.*



Given the angle AOB.

Required to bisect $\angle AOB$.

Construction. With O as a center and any radius describe an arc cutting OA at X and OB at Y. Post. 4

With X and Y as centers and XY as a radius describe arcs intersecting at P. Post. 4

Draw OP. Post. 1

Then OP bisects $\angle AOB$. Q.E.F.

Proof. Draw PX and PY.

Then prove that the $\triangle OXP$ and OYP are congruent. § 80

Then $\angle AOP = \angle POB$, by § 67. Q.E.D.

EXERCISE 35

1. To construct an angle of 45°; of 135°.

2. To construct an angle of 22° 30′; of 157° 30′.

3. To construct an equilateral triangle, having given one side, and thus to construct an angle of 60°.

4. To construct an angle of 30°; and thus to trisect a right angle.

5. To construct an angle of 15°; of 7° 30′; of 195°; of 345°.

6. To construct a triangle having two of its angles equal to 75°. Is the triangle definitely determined?

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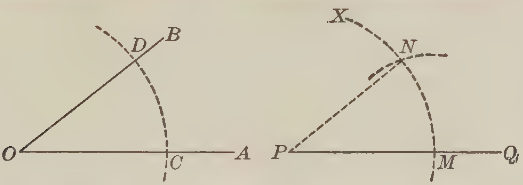
確定之名稱

Proposition XXV. Problem

232. *From a given point in a given line, to draw a line making an angle equal to a given angle.*



Given the angle *AOB* and the point *P* in the line *PQ*.

Required from P to draw a line making with the line PQ an angle equal to $\angle AOB$.

Construction. With O as a center and any radius describe an arc cutting OA at C and OB at D. Post. 4

With P as a center and the same radius describe an arc MX, cutting PQ at M. Post. 4

With M as a center and a line joining C and D as a radius describe an arc cutting the arc MX at N. Post. 4

Draw PN. Post. 1

Then $\angle QPN = \angle AOB$. Q. E. F.

Proof. Draw CD and MN.

Then prove that the $\triangle PMN$ and OCD are congruent. § 80

Then $\angle QPN = \angle AOB$, by § 67. Q. E. D.

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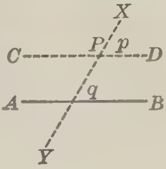
233. Corollary. *Through a given external point, to draw a line parallel to a given line.*



Let AB be the given line and P the given external point.

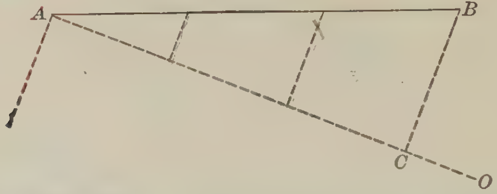
Draw any line XPY through P, cutting AB as in the figure.

Draw CD through P, making $\angle p = \angle q$.

The line CD will be the line required.

PROPOSITION XXVI. PROBLEM

234. *To divide a given line into a given number of equal parts.*



Given the line *AB*.

Required to divide AB into a given number of equal parts.

Construction. From A draw the line AO, making any convenient angle with AB. Post. 1

Take any convenient length, and by describing arcs apply it to AO as many times as is indicated by the number of parts into which AB is to be divided. Post. 4

From C, the last point thus found on AO, draw CB. Post. 1

From the division points on AO draw parallels to CB. § 233

These lines divide AB into equal parts. Q.E.F.

Proof. These lines divide AB into equal parts, by § 134. Q.E.D.

EXERCISE 36

1. To divide a given line into four equal parts.

2. To construct an equilateral triangle, given the perimeter.

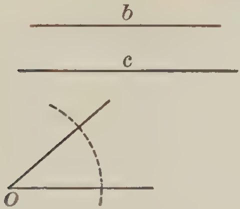
3. Through a given point, to draw a line which shall make equal angles with the two sides of a given angle.

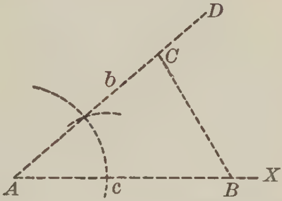
4. Through a given point, to draw two lines so that they shall form with two intersecting lines two isosceles triangles.

5. To construct a triangle having its three angles respectively equal to the three angles of a given triangle.

PROPOSITION XXVII. PROBLEM

235. *To construct a triangle when two sides and the included angle are given.*





Given b and c two sides of a triangle, and O the included angle.

Required to construct the triangle.

Construction. On any line as AX, by describing an arc, mark off AB equal to c. Post. 4

At A construct $\angle BAD$ equal to $\angle O$. § 232

On AD, by describing an arc, mark off AC equal to b. Post. 4

Draw BC. Post. 1

Then $\triangle ABC$ is the $\triangle$ required. Q.E.F.

Proof. (Left for the student.)

236. COROLLARY 1. *To construct a triangle when a side and two angles are given.*

There are two cases to be considered: (1) when the given side is included between the given angles; and (2) when it is not (in which case find the other angle by § 107).

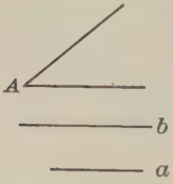
237. COROLLARY 2. *To construct a triangle when the three sides are given.*

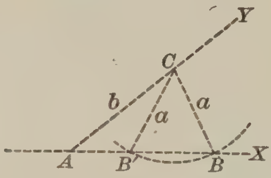
238. COROLLARY 3. *To construct a parallelogram when two sides and the included angle are given.*

Combine § 235 and § 233.

PROPOSITION XXVIII. PROBLEM

239. *To construct a triangle when two sides and the angle opposite one of them are given.*





Given *a* **and** *b* **two sides of a triangle, and** *A* **the angle opposite** *a*.

Required to construct the triangle.

Construction. CASE 1. *If a is less than b.*

Construct $\angle XAY$ equal to the given $\angle A$. § 232

On AY take AC equal to b.

From C as a center, with a radius equal to a, describe an arc intersecting the line AX at B and B'.

Draw BC and $B'C$, thus completing the triangle.

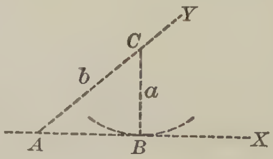
Then both the $\triangle ABC$ and $AB'C$ satisfy the conditions, and hence we have two constructions. Q.E.F.

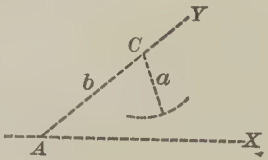
This is called the *ambiguous* case.

Discussion. If the given side a is equal to the $\perp CB$, the arc described from C will touch AX, and there will be but one construction, the rt. $\triangle ABC$.

If the given side a is less than the perpendicular from C, the arc described from C will not intersect or touch AX, and hence a construction is impossible.

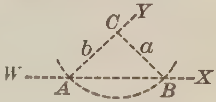
If $\angle A$ is right or obtuse, a construction is impossible, since $a < b$; for the side of a triangle opposite a right or obtuse angle is the longest side (§ 114).





Case 2. *If a is equal to b.*

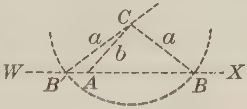
If the given $\angle A$ is acute, and $a = b$, the arc described from C as a center, and with a radius equal to a, will cut the line WX at the points A and B. There is therefore but one triangle that satisfies the conditions, namely the isosceles $\triangle ABC$.



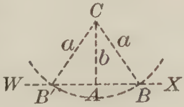
Discussion. If the $\angle A$ is right or obtuse, a construction is impossible when $a = b$; for equal sides of a triangle have equal angles opposite them, and a triangle cannot have two right angles or two obtuse angles (§ 109).

Case 3. *If a is greater than b.*

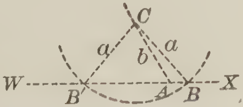
If the given $\angle A$ is acute, the arc described from C will cut the line WX on opposite sides of A, at B and B'. The $\triangle ABC$ satisfies the conditions, but the $\triangle AB'C$ does not, for it does not contain the acute $\angle A$. There is then only one triangle that satisfies the conditions, namely the $\triangle ABC$.



If the given $\angle A$ is right, the arc described from C cuts the line WX on opposite sides of A at the points B and B', and we have the two *congruent* right triangles ABC and $AB'C$ that satisfy the conditions.



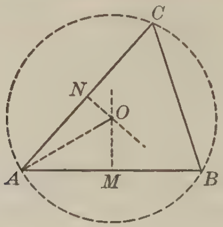
If the given $\angle A$ is obtuse, the arc described from C cuts the line WX on opposite sides of A, at the points B and B'. The $\triangle ABC$ satisfies the conditions, but the $\triangle AB'C$ does not, for it does not contain the obtuse $\angle A$. There is then only one triangle that satisfies the conditions, namely the $\triangle ABC$.



Discussion. We therefore see that when $a > b$, we have only one triangle that satisfies the conditions, for the two congruent right triangles give us only one distinct triangle.

Proposition XXIX. Problem

240. *To circumscribe a circle about a given triangle.*



Given the triangle *ABC*.

Required to circumscribe a ⊙ about △ ABC.

Construction. Draw the perpendicular bisectors of the sides *AB* and *AC*. § 229

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Since *AB* is not the prolongation of *CA*, these ⊥s will intersect at some point *O*. Otherwise they would be ∥, and one of them would have to be ⊥ to two intersecting lines. § 82

With *O* as a center, and a radius *OA*, describe a circle. Post. 4

The ⊙ *ABC* is the ⊙ required. Q.E.F.

Proof. The point *O* is equidistant from *A* and *B*, and also is equidistant from *A* and *C*. § 150

∴ the point *O* is equidistant from *A*, *B*, and *C*.

∴ a ⊙ described with *O* as a center, with a radius equal to *OA*, will pass through the vertices *A*, *B*, and *C*, by § 160. Q.E.D.

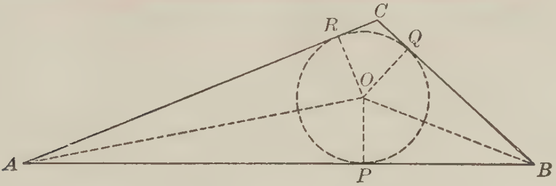
241. Corollary 1. *To describe a circle through three points not in the same straight line.*

242. Corollary 2. *To find the center of a given circle or of the circle of which an arc is given.*

243. **Circumcenter.** The center of the circle circumscribed about a polygon is called the *circumcenter* of the polygon.

PROPOSITION XXX. PROBLEM

244 *To inscribe a circle in a given triangle.*



Given the triangle *ABC*.

Required to inscribe a ⊙ in △ ABC.

Construction. Bisect the ∠s *A* and *B*. § 231

From *O*, the intersection of the bisectors, draw *OP* ⊥ to the side *AB*. § 227

With *O* as a center and a radius *OP*, describe the ⊙ *PQR*.

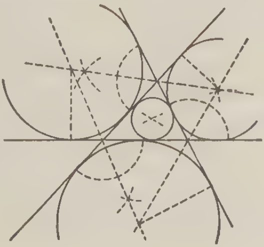
The ⊙ *PQR* is the ⊙ required. Q.E.F.

Proof. Since *O* is in the bisector of the ∠ *A*, it is equidistant from the sides *AB* and *AC*; and since *O* is in the bisector of the ∠ *B*, it is equidistant from the sides *AB* and *BC*. § 152

∴ a circle described with *O* as a center, and a radius *OP*, will touch the sides of the triangle, by § 184. Q.E.D.

245. Incenters and Excenters. The center of a circle inscribed in a polygon is called the *incenter* of the polygon.

The intersections of the bisectors of the exterior angles of a triangle are the centers of three circles, each tangent to one side of the triangle and the two other sides produced. These three circles are called *escribed* circles; and their centers are called the *excenters* of the triangle.



Proposition XXXI. Problem

246. *Through a given point, to draw a tangent to a given circle.*

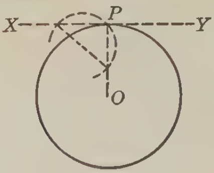


Fig. 1

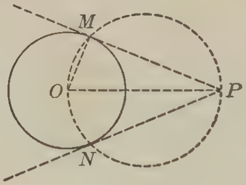


Fig. 2

Given the point P and the circle with center O.

Required through P to draw a tangent to the circle.

Case 1. *When the given point is on the circle* (Fig. 1).

Construction. From the center O draw the radius OP. Post. 1

Through P draw $XY \perp$ to OP. § 228

Then XY is the tangent required. Q.E.F.

Proof. Since XY is $\perp$ to the radius OP, Const.

$\therefore XY$ is tangent to the $\odot$ at P, by § 184. Q.E.D.

Case 2. *When the given point is outside the circle* (Fig. 2).

Construction. Draw OP. Post. 1

Bisect OP. § 229

With the mid-point of OP as a center and a radius equal to $\frac{1}{2}OP$, describe a circle intersecting the given circle at the points M and N, and draw PM.

Then PM is the tangent required. Q.E.F.

Proof. Draw OM.

$\angle OMP$ is a right angle. § 215

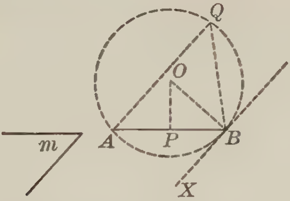
$\therefore PM$ is $\perp$ to OM. § 27

$\therefore PM$ is tangent to the circle at M, by § 184. Q.E.D.

Discussion. In like manner, we may prove PN tangent to the given $\odot$.

PROPOSITION XXXII. PROBLEM

247. *Upon a given line as a chord, to describe a segment of a circle in which a given angle may be inscribed.*



Given the line *AB* and the angle *m*.

Required on AB as a chord, to describe a segment of a circle in which ∠ m may be inscribed.

Construction. Construct the ∠ ABX equal to the ∠ m. § 232

Bisect the line AB by the ⊥ PO. § 229

From the point B draw BO ⊥ to XB. § 228

With O, the point of intersection of PO and BO, as a center, and a radius equal to OB, describe a circle.

The segment BQA is the segment required. Q.E.F.

Proof. The point O is equidistant from A and B. § 150

∴ the circle will pass through A and B. § 160

But BX is ⊥ to OB. Const.

∴ BX is tangent to the ⊙. § 184

∴ ∠ ABX is measured by $\frac{1}{2}$ arc AB. § 220

But any angle, as the ∠ Q, inscribed in the segment BQA is measured by $\frac{1}{2}$ arc AB. § 214

∴ ∠ Q = ∠ ABX. Ax. 8

But ∠ ABX = ∠ m. Const.

∴ ∠ m may be inscribed in the segment BQA, by § 217. Q.E.D

248. How to attack a Problem. There are three common methods by which to attack a new problem:

(1) By synthesis; 组合 综合 配合
(2) By analysis; 分析 分解 纲目 要略
(3) By the intersection of loci.

249. Synthetic Method. If a problem is so simple that the solution is obvious from a known proposition, we have only to make the construction according to the proposition, and then to give the synthetic proof, if a proof is necessary, that the construction is correct.

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It is rarely the case, however, that a problem is so simple as to allow this method to be used. We therefore commonly resort at once to the second method.

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250. Analytic Method. This is the usual method of attack, and is as follows:

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(1) Suppose the problem solved and see what results follow.

(2) Then see if it is possible to attain these results and thus effect the required construction; in other words, try to work backwards.

The third method, by the intersection of loci, is considered on page 143.

251. Determinate, Indeterminate, and Impossible Cases. A problem that has a definite number of solutions is said to be *determinate*. A problem that has an indefinite number of solutions is said to be *indeterminate*. A problem that has no solution is said to be *impossible*.

For example, to construct a triangle, having given its sides, is determinate; to construct a quadrilateral, having given its sides, is indeterminate; to construct a triangle with sides 2 in., 3 in., and 6 in. is impossible

252. Discussion. The examination of a problem with reference to all possible conditions, particularly with respect to the number of solutions, is called the *discussion* of the problem.

Discussions have been given in several of the preceding problems.

253. Applications of the Analytic Method. The following are examples of the use of analysis in the solution of problems.

EXERCISE 37

1. In a triangle ABC, to draw PQ parallel to the base AB, cutting the sides in P and Q, so that PQ shall equal $AP + BQ$.

Analysis. Assume the problem solved.

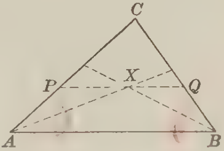
Then AP must equal some part of PQ, as PX, and BQ must equal QX.

But if $AP = PX$, what must $\angle PXA$ equal?

$\because PQ$ is $\parallel$ to AB, what does $\angle PXA$ equal?

Then why must $\angle BAX = \angle XAP$?

Similarly, what about $\angle QBX$ and $\angle XBA$?



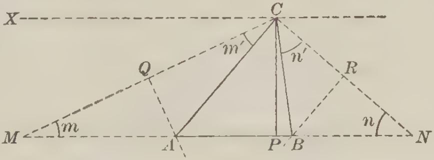
Construction. Now reverse the process. What should we do to $\angle$s A and B in order to fix X? Then how shall PQ be drawn? Now give the proof.

2. To construct a triangle, having given the perimeter, one angle, and the altitude from the vertex of the given angle.

Analysis. Let ABC be the triangle, $\angle C$ the given angle, and CP the given altitude, and assume that the problem is solved.

Since the perimeter is given as a definite line, we now try producing AB and BA, making $BN = BC$, and $AM = AC$.

Then $\angle m =$ what angle, and $\angle n =$ what angle?



Then $\angle m + \angle n + \angle MCN = 180°$.

But $\angle MCN = \angle m' + \angle ACB + \angle n'$.

$$\therefore 2\angle m + 2\angle n + \angle ACB = 180°. \text{ (Why?)}$$

$$\therefore \angle m + \angle n + \tfrac{1}{2}\angle ACB = 90°,$$

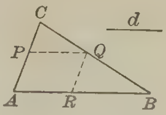
or $$\angle m + \angle n = 90° - \tfrac{1}{2}\angle ACB.$$

$$\therefore \angle MCN = 90° + \tfrac{1}{2}\angle ACB. \text{ (Why?)}$$

$$\therefore \angle MCN \text{ is known.}$$

Construction. Now reverse the process. Draw MN equal to the perimeter. Then on MN construct a segment in which $\angle MCN$ may be inscribed (§ 247). Draw $XC \parallel$ to MN at the distance CP from MN, cutting the arc at C. Then A and B are on the $\perp$ bisectors of CM and CN. Why?

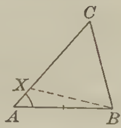
3. To draw through two sides of a triangle a line parallel to the third side, so that the part intercepted between the sides shall have a given length.



If $PQ = d$, what does AR equal? How will you reverse the reasoning?

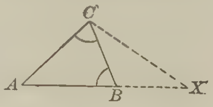
4. To draw a tangent to a given circle so that it shall be parallel to a given line.

5. To construct a triangle, having given a side, an adjacent angle, and the difference of the other sides.



If AB, $\angle A$, and $AC - BC$ are known, what points are determined? Then can XB be drawn? What kind of a triangle is $\triangle XBC$? How can C be located?

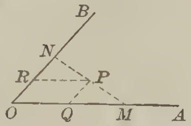
6. To construct a triangle, having given two angles and the sum of two sides.



Can the third $\angle$ be found? Assume the problem solved. If $AX = AB + BC$, what kind of a triangle is $\triangle BXC$? What does $\angle CBA$ equal? Is $\angle X$ known? How can C be fixed?

7. To construct a square, having given the diagonal.

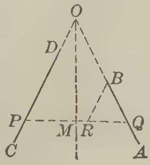
8. To draw through a given point P between the sides of an angle AOB a line terminated by the sides of the angle and bisected at P.



If $PM = PN$, and PR is $\parallel$ to AO, what can you say as to OR and RN? Can you now reverse this? Similarly, if PQ is $\parallel$ to BO, is $OQ =$ to QM?

9. To draw a line that would bisect the angle formed by two lines if those lines were produced to meet.

If AB and CD are the given lines, consider what would be the conditions if they could be produced to meet at O. Then the bisector of $\angle O$ would be the $\perp$ bisector of PQ, a line drawn so as to make equal angles with the two given lines.

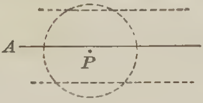


Now reverse this. How can we draw PQ so as to make $\angle P = \angle Q$? Draw $BR \parallel$ to DC, and lay off $BR = BQ$. Then draw QRP and prove that this is such a line. Then draw its $\perp$ bisector.

254. Intersection of Loci. The third general method of attack mentioned in § 248 is by intersection of loci. This is very convenient when we wish to find a point satisfying two conditions, each of which involves some locus.

EXERCISE 38

1. To find a point that is $\frac{1}{4}$ in. from a given point and $\frac{3}{16}$ in. from a given line.



If P is the given point, what is the locus of a point $\frac{1}{4}$ in. from P? If AB is the given line, what is the locus of a point $\frac{3}{16}$ in. from AB? These two loci intersect in how many points at most? Discuss the solution.

2. To find a point that is $\frac{1}{2}$ in. from one given point and $\frac{3}{4}$ in. from another given point.

Discuss the number of possible points answering the conditions.

3. To find a point that is $\frac{1}{2}$ in. from the vertex of an angle and equidistant from the sides of the angle.

4. To find a point that is equidistant from two intersecting lines and $\frac{1}{4}$ in. from their point of intersection.

How many such points can always be found?

5. To find a point that is $\frac{1}{2}$ in. from a given point and equidistant from two intersecting lines.

Discuss the problem for various positions of the given point.

6. To find a point that is $\frac{1}{2}$ in. from a given point and equidistant from two parallel lines.

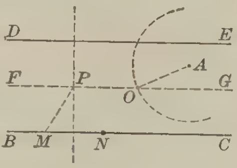
Discuss the problem for various positions of the given point.

7. Find the locus of the mid-point of a chord of a given length that can be drawn in a given circle.

8. Find the locus of the mid-point of a chord drawn through a given point within a given circle.

9. To describe a circle that shall pass through a given point and cut equal chords of a given length from two parallels.

Analysis. Let A be the given point, BC and DE the given parallels, MN the given length, and O the center of the required circle.



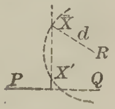
Since the circle cuts equal chords from two parallels, what must be the relative distance of its center from each? Therefore what line must be one locus for O?

Draw the ⊥ bisector of MN, cutting FG at P. How, then, does PM compare with the radius of the circle required? How shall we then find a point O on FG that is at a distance PM from A? Do we then know that O is the center of the required circle?

10. To describe a circle that shall be tangent to each of two given intersecting lines.

11. To find in a given line a point that is equidistant from two given points.

12. To find a point that is equidistant from two given points and at a given distance from a third given point.



13. To describe a circle that has a given radius and passes through two given points.

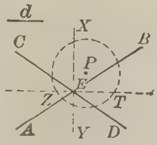
14. To find a point at given distances from two given points.

15. To describe a circle that has its center in a given line and passes through two given points.

16. To find a point that is equidistant from two given points and also equidistant from two given intersecting lines.

17. To find a point that is equidistant from two given points and also equidistant from two given parallel lines.

18. To find a point that is equidistant from two given intersecting lines and at a given distance from a given point.



19. To find a point that lies in one side of a given triangle and is equidistant from the other two sides.

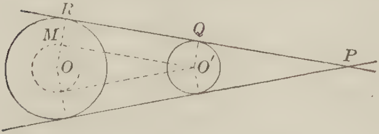
255. General Directions for solving Problems. In attacking a new problem draw the most general figure possible and the solution may be evident at once. If the solution is not evident, see if it depends on finding a point, in which case see if two loci can be found. If this is not the case, assume the problem solved and try to work backwards,—the method of analysis.

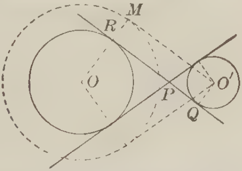
EXERCISE 39

1. To draw a common tangent to two given circles.

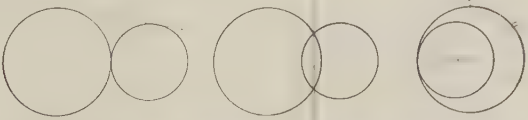
If the centers are O and O' and the radii r and r', the tangent QR seems to be ∥ to $O'M$, a tangent from O' to a circle whose radius is $r - r'$. If this is true, we can easily reverse the process. Since there are two tangents from O', so there are two common tangents.

In the right-hand figure the tangent QR seems to be ∥ to $O'M$, a tangent from O' to a circle whose radius is $r + r'$. If this is true, we can easily reverse the process. There are four common tangents in general.





2. To draw a common tangent to two given circles, using the following figures.



3. The locus of the vertex of a right triangle, having a given hypotenuse as its base, is the circle described upon the given hypotenuse as a diameter.

4. The locus of the vertex of a triangle, having a given base and a given angle at the vertex, is the arc which forms with the base a segment in which the given angle may be inscribed.

To construct an isosceles triangle, having given:

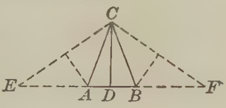
5. The base and the angle at the vertex.

6. The base and the radius of the circumscribed circle.

7. The base and the radius of the inscribed circle.

8. The perimeter and the altitude.

Let ABC be the $\triangle$ required, EF the given perimeter. The altitude CD passes through the middle of EF, and the $\triangle$s EAC, BFC are isosceles.



To construct a right triangle, having given:

9. The hypotenuse and one side.

10. One side and the altitude upon the hypotenuse.

11. The median and the altitude upon the hypotenuse.

12. The hypotenuse and the altitude upon the hypotenuse.

13. The radius of the inscribed circle and one side.

14. The radius of the inscribed circle and an acute angle.

To construct a triangle, having given:

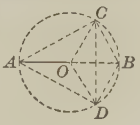
15. The base, the altitude, and an angle at the base.

16. The base, the altitude, and the angle at the vertex.

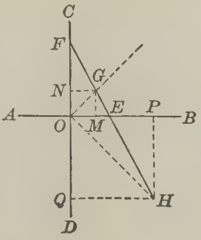
17. One side, an adjacent angle, and the sum of the other sides.

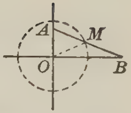
18. To construct an equilateral triangle, having given the radius of the circumscribed circle.

19. To construct a rectangle, having given one side and the angle between the diagonals.

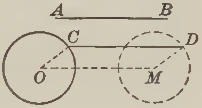


20. Given two perpendiculars, AB and CD, intersecting in O, and a line intersecting these perpendiculars in E and F; to construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie in EF. (Two solutions.)

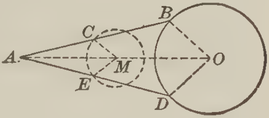




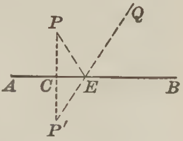
21. A straight rod moves so that its ends constantly touch two fixed rods perpendicular to each other. Find the locus of its mid-point.



22. A line moves so that it remains parallel to a given line, and so that one end lies on a given circle. Find the locus of the other end.



23. Find the locus of the mid-point of a line-segment that is drawn from a given external point to a given circle.

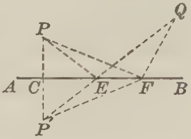


24. To draw lines from two given points P and Q which shall meet on a given line AB and make equal angles with AB.

$\because \angle BEQ = \angle PEC$, $\therefore \angle CEP' = \angle PEC$. (Why?)

But it is easy to make $\angle CEP' = \angle PEC$, by making $PP' \perp AB$, and $CP' = PC$, and joining P' and Q.

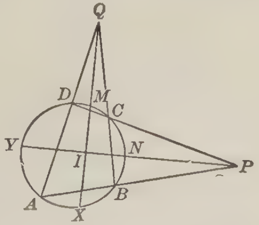
25. To find the shortest path from a point P to a line AB and thence to a point Q.



Prove that $PE + EQ < PF + FQ$, where $\angle BEQ = \angle PEC$.

This shows that a ray of light from a point to a plane mirror and thence to another point takes the shortest possible path.

26. The bisectors of the angles included by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.



Arc AX – arc MD

$= \text{arc } XB - \text{arc } CM$. (Why?)

Arc YA – arc BN

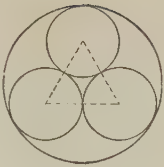
$= \text{arc } DY - \text{arc } NC$. (Why?)

$\therefore$ arc YX + arc NM

$= \text{arc } MY + \text{arc } XN$. (Why?)

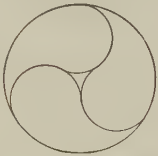
$\therefore \angle YIX = \angle XIN$. (Why?)

How does this prove the proposition? Discuss the impossible case.



27. Construct this design, making the figure twice this size.

Construct the equilateral △. Then describe the small ⓢ with half the side of the △ as a radius. Then find the radius of the circumscribing ⊙.

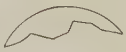


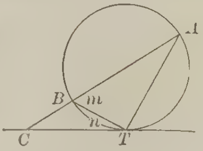
28. A circular window in a church has a design similar to the accompanying figure. Draw it, making the figure twice this size.

This is made from the figure of the preceding exercise, by erasing certain lines.

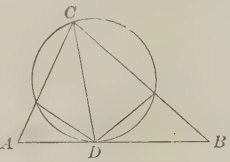
29. Two wheels of radii 1 ft. 6 in. and 2 ft. 3 in. respectively are connected by a belt, drawn straight between the points of tangency. The centers being 6 ft. apart, draw the figure mathematically. Use the scale of 1 in. to the foot.

30. A water wheel is broken and all but a fragment is lost. A workman wishes to restore the wheel. Make a drawing showing how he can construct a wheel the size of the original.



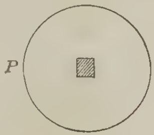


31. In this figure $\angle m = 62^\circ$, and $\angle n = 28^\circ$. Find the number of degrees in each of the other angles, and determine whether AB is a diameter.



32. In this figure $\angle B = 41^\circ$, $\angle A = 65^\circ$, and $\angle BDC = 97^\circ$. Find the number of degrees in each of the other angles, and determine whether CD is a diameter.

33. Construct or explain why it is impossible to construct a triangle with sides 3 in., 2 in., 6 in.; also one with sides 5 in., 7 in., 12 in.; also one with sides 2 in., 1 in., $1\frac{1}{2}$ in.



34. Show how to draw a tangent to this circle at the point P, the center of the circle not being accessible.

EXERCISE 40

1. In a circle whose center is O the chord AB is drawn so that $\angle BAO = 27°$. How many degrees are there in $\angle AOB$?

2. In a circle whose center is O the chord AB is drawn so that $\angle BAO = 25°$. On the circle, and on the same side of AB as the center O, the point D is taken and is joined to A and B. How many degrees are there in $\angle ADB$?

3. What is the locus of the mid-point of a chord of a circle formed by secants drawn from a given external point?

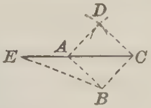
4. In a circle whose center is O two perpendiculars OM and ON are drawn to the chords AB and CD respectively, and it is known that $\angle NMO = \angle ONM$. Prove that $AB = CD$.

5. Two circles intersect at the points A and B. Through A a variable secant is drawn, cutting the circles at C and D. Prove that the angle DBC is constant.

6. Let A and B be two fixed points on a given circle, and M and N be the extremities of a rotating diameter of the same circle. Find the locus of the point of intersection of the lines AM and BN.

7. Upon a line AB a segment of a circle containing 240° is constructed, and in the segment any chord PQ subtending an arc of 60° is drawn. Find the locus of the point of intersection of AP and BQ; also of AQ and BP.

8. To construct a square, given the sum of the diagonal and one side.



Let $ABCD$ be the square required, and CA the diagonal. Produce CA, making $AE = AB$. $\triangle ABC$ and ABE are isosceles and $\angle BAC = \angle ACB = 45°$. Find the value of $\angle E$. Construct $\angle CBE$. Now reverse the reasoning.

The propositions in Exercise 40 are taken from recent college entrance examination papers.

EXERCISE 41

Review Questions

1. Define the word *circle* and the principal terms used in connection with it.

2. What is meant by a central angle? How is it measured?

3. What is meant by an inscribed angle? How is it measured?

4. State the general proposition covering all the cases that have been considered relating to the measure of an angle formed by the intersection of two secants.

5. State all of the facts you have learned relating to equal chords of a circle.

6. State all of the facts you have learned relating to unequal chords of a circle.

7. State all of the facts you have learned relating to tangents to a circle.

8. How many points are required to determine a straight line? two parallel lines? an angle? a circle?

9. Name one kind of magnitude that you have learned to trisect, and state how you proceed to trisect this magnitude.

10. In order to construct a definite triangle, what parts must be known?

11. What are the important methods of attacking a new problem in geometry? Which is the best method to try first?

12. What is meant by determinate, indeterminate, and impossible cases in the solution of a problem?

13. Distinguish between a constant and a variable, and give an illustration of each.

14. Distinguish between inscribed, circumscribed, and escribed circles.

15. What is meant by the statement that a central angle is measured by the intercepted arc?

BOOK III

PROPORTION. SIMILAR POLYGONS

256. Proportion. An expression of equality between two equal ratios is called a *proportion.*

257. Symbols. A proportion is written in one of the following forms: $\frac{a}{b}=\frac{c}{d}$; $a:b=c:d$; $a:b::c:d$.

This proportion is read "a is to b as c is to d"; or "the ratio of a to b is equal to the ratio of c to d."

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258. Terms. In a proportion the four quantities compared are called the *terms.* The first and third terms are called the *antecedents;* the second and fourth terms, the *consequents.* The first and fourth terms are called the *extremes;* the second and third terms, the *means.* 平均的 不極端的外項外率

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Thus in the proportion $a:b=c:d$, a and c are the antecedents, b and d the consequents, a and d the extremes, b and c the means.

259. Fourth Proportional. The fourth term of a proportion is called the *fourth proportional* to the terms taken in order.

Thus in the proportion $a:b=c:d$, d is the fourth proportional to a, b, and c.

260. Continued Proportion. The quantities a, b, c, d, $\cdots$ are said to be in *continued proportion,* if $a:b=b:c=c:d=\cdots$.

If three quantities are in continued proportion, the second is called the *mean proportional* between the other two, and the third is called the *third proportional* to the other two.

Thus in the proportion $a:b=b:c$, b is the mean proportional betweer a and c, and c is the third proportional to a and b.

PROPOSITION I. THEOREM

261. *In any proportion the product of the extremes is equal to the product of the means.*

Given $a : b = c : d.$

To prove that $ad = bc.$

Proof. $\frac{a}{b} = \frac{c}{d}.$ § 257

Multiplying by bd, $ad = bc$, by Ax. 3. Q.E.D.

262. COROLLARY 1. *The mean proportional between two quantities is equal to the square root of their product.*

For if $a : b = b : c$, then $b^2 = ac$ (§ 261), and $b = \sqrt{ac}$, by Ax. 5.

263. COROLLARY 2. *If the two antecedents of a proportion are equal, the two consequents are equal.*

264. COROLLARY 3. *If the product of two quantities is equal to the product of two others, either two may be made the extremes of a proportion in which the other two are made the means.*

For if $ad = bc$, then, by dividing by bd, $\frac{a}{b} = \frac{c}{d}$, by Ax. 4.

PROPOSITION II. THEOREM

265. *If four quantities are in proportion, they are in proportion by alternation; that is, the first term is to the third as the second term is to the fourth.*

Given $a : b = c : d.$

To prove that $a : c = b : d.$

Proof. $ad = bc.$ § 261

Dividing by cd, $\frac{a}{c} = \frac{b}{d},$ Ax. 4

or $a : c = b : d$, by § 257. Q.E.D.

PROPOSITION III. THEOREM

266. *If four quantities are in proportion, they are in proportion by inversion; that is, the second term is to the first as the fourth term is to the third.*

Given $a : b = c : d.$

To prove that $b : a = d : c.$

Proof. $bc = ad.$ § 261

Dividing each member of the equation by ac,

$$\frac{b}{a} = \frac{d}{c}, \qquad \text{Ax. 4}$$

or $b : a = d : c$, by § 257. Q.E.D.

PROPOSITION IV. THEOREM

267. *If four quantities are in proportion, they are in proportion by composition; that is, the sum of the first two terms is to the second term as the sum of the last two terms is to the fourth term.*

Given $a : b = c : d.$

To prove that $a + b : b = c + d : d.$

Proof. $\frac{a}{b} = \frac{c}{d}.$ § 257

Adding 1 to each member of the equation,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1, \qquad \text{Ax. 1}$$

or

$$\frac{a+b}{b} = \frac{c+d}{d}.$$

$\therefore a + b : b = c + d : d$, by § 257. Q.E.D.

In a similar manner it may be shown that

$$a + b : a = c + d : c.$$

PROPOSITION V. THEOREM

268. *If four quantities are in proportion, they are in proportion by division; that is, the difference of the first two terms is to the second term as the difference of the last two terms is to the fourth term.*

Given $a : b = c : d.$

To prove that $a - b : b = c - d : d.$

Proof. $\frac{a}{b} = \frac{c}{d}.$ § 257

$\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1,$ Ax. 2

or $\frac{a-b}{b} = \frac{c-d}{d}.$

$\therefore a - b : b = c - d : d$, by § 257. Q.E.D.

In a similar manner it may be shown that

$$a - b : a = c - d : c.$$

PROPOSITION VI. THEOREM

269. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Given $a : b = c : d = e : f = g : h.$

To prove that $a + c + e + g : b + d + f + h = a : b.$

Proof. Let $r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}.$

Then $a = br, \quad c = dr, \quad e = fr, \quad g = hr.$ Ax. 3

$\therefore a + c + e + g = (b + d + f + h)r.$ Ax. 1

$\therefore \frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b}.$ Ax. 4

$\therefore a + c + e + g : b + d + f + h = a : b$, by § 257. Q.E.D.

PROPOSITION VII. THEOREM

270. *Like powers of the terms of a proportion are in proportion.*

Given $a : b = c : d.$

To prove that $a^n : b^n = c^n : d^n.$

Proof. $\frac{a}{b} = \frac{c}{d}.$ § 257

$\therefore \frac{a^n}{b^n} = \frac{c^n}{d^n}$, by Ax. 5. Q.E.D.

PROPOSITION VIII. THEOREM

271. *If three quantities are in continued proportion, the first is to the third as the square of the first is to the square of the second.*

Given $a : b = b : c.$

To prove that $a : c = a^2 : b^2.$

Proof. $a^2 = a^2,$ Iden.

and $ac = b^2.$ § 261

$\therefore \frac{a^2}{ac}$ or $\frac{a}{c} = \frac{a^2}{b^2}.$ Ax. 4

$\therefore a : c = a^2 : b^2$, by § 257. Q.E.D.

272. Nature of the Quantities in a Proportion. Although we may have ratios of lines, or of areas, or of solids, or of angles, we treat all of the terms of a proportion as *numbers*.

If b and d are lines or solids, for example, we cannot multiply each member of $\frac{a}{b} = \frac{c}{d}$ by bd, as in § 261.

Hence *when we speak of the product of two geometric magnitudes, we mean the product of the numbers that represent them when expressed in terms of a common unit.*

EXERCISE 42

1. Prove that $a:b=ma:mb$.

2. If $a:b=c:d$, and $m:n=p:q$, prove that $am:bn=cp:dq$.

If $a:b=c:d$, *prove the following:*

3. $a:d=bc:d^2$.
4. $1:b=c:ad$.
5. $ad:b=c:1$.
6. $ma:b=mc:d$.
7. $ma:nb=mc:nd$.
8. $a-1:b=bc-d:bd$.
9. $a+1:1=bc+d:d$.
10. $1:bc=1:ad$.
11. $a+b:a-b=c+d:c-d$.

In Ex. 11, use § 267 and § 268, and Ax. 4. In this case a, b, c, and d are said to be in proportion *by composition and division.*

If $a:b=b:c$, *prove the following:*

12. $c:b=b:a$.
13. $a:c=b^2:c^2$.
14. $(b+\sqrt{ac})(b-\sqrt{ac})=0$
15. $ac-1:b-1=b+1:1$.

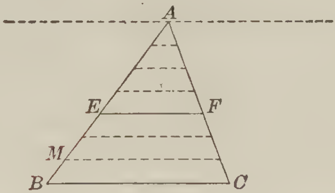
16. If $2:7=3:x$, show that $2x=21$, and $x=10\frac{1}{2}$.

Find the value of x *in the following:*

17. $1:7=3:x$.
18. $2:9=5:x$.
19. $4:28=3:x$.
20. $2:8=x:12$.
21. $3:5=x:9$.
22. $7:21=x:5$.
23. $3:5=x+1:10$.
24. $8:15=2x+3:45$.
25. $0.8:x=4:9$.
26. $0.7:x=21:15$.
27. $0.25:x=5:8$.
28. $x:1.3=4:0.26$.
29. $x:2.7=7:5.4$.
30. $x:8.1=0.3:0.9$.
31. $2:x=x:32$.
32. $7:x=x:28$.
33. $1:1+x=x-1:3$.
34. $5:x-2=x+2:1$.
35. $x^2:2a=3a:6$.
36. $x:4a=2a^2:x^2$.
37. $a:1=x-1:7$.
38. $x+1:x-1=3:2$.
39. $3:x+4=x-4:3$.
40. $ab:b=b-cx:bc-x$.

PROPOSITION IX. THEOREM

273. *If a line is drawn through two sides of a triangle parallel to the third side, it divides the two sides proportionally.*



Given the triangle ABC, with EF drawn parallel to BC.

To prove that $EB : AE = FC : AF$.

CASE 1. *When AE and EB are commensurable.*

Proof. Assume that MB is a common measure of AE and EB. Let MB be contained m times in EB, and n times in AE.

Then $EB : AE = m : n$.

(*For m and n are the numerical measures of EB and AE.*)

At the points of division on EB and AE draw lines ∥ to BC. These lines will divide AC into $m + n$ equal parts, of which FC will contain m parts, and AF will contain n parts. § 134

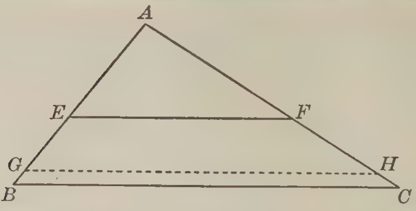
$$\therefore FC : AF = m : n.$$

$$\therefore EB : AE = FC : AF, \text{ by Ax. 8.}$$ Q.E.D.

For practical purposes this proves the proposition, for even if AE and EB are incommensurable, we can, by taking a unit of measure small enough, find the measure of AE and EB to as close a degree of approximation as we may desire, just as we can carry $\sqrt{2}$ to as many decimal places as we wish, although its exact value cannot be expressed rationally.

On this account many teachers omit the incommensurable case discussed on page 158, or merely require the proof there given to be read aloud and explained by the class.

CASE 2. *When AE and EB are incommensurable.*



Proof. Divide AE into a number of equal parts, and apply one of these parts to EB as many times as EB will contain it.

Since AE and EB are incommensurable, a certain number of these parts will extend from E to some point G, leaving a remainder GB less than one of these parts.

Draw $GH \parallel$ to BC.

Then $$EG : AE = FH : AF. \qquad \text{Case 1}$$

By increasing the *number* of equal parts into which AE is divided, we can make the *length* of each part less than any assigned positive value, however small, but not zero.

Hence GB, which is less than one of these equal parts, has zero for a limit. § 204

And the corresponding segment HC has zero for a limit.

Therefore EG approaches EB as a limit,

and FH approaches FC as a limit.

$$\therefore \text{ the variable } \frac{EG}{AE} \text{ approaches } \frac{EB}{AE} \text{ as a limit,}$$

$$\text{and the variable } \frac{FH}{AF} \text{ approaches } \frac{FC}{AF} \text{ as a limit.}$$

$$\text{But } \frac{EG}{AE} \text{ is always equal to } \frac{FH}{AF}. \qquad \text{Case 1}$$

$$\therefore \frac{EB}{AE} = \frac{FC}{AF}, \text{ by § 207.}$$

Q.E.D.

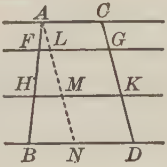
274. Corollary 1. *One side of a triangle is to either of its segments cut off by a line parallel to the base as the third side is to its corresponding segment.*

For $EB:AE = FC:AF.$ § 273

By composition, $EB + AE:AE = FC + AF:AF,$ § 267

or $AB:AE = AC:AF.$ Ax. 11

275. Corollary 2. *Three or more parallel lines cut off proportional intercepts on any two transversals.*



Draw $AN \parallel$ to CD.

Then $AL = CG$, $LM = GK$, $MN = KD$. § 127

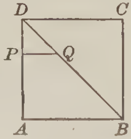
Now $AH:AM = AF:AL = FH:LM$, § 274

and $AH:AM = HB:MN.$ § 273

$\therefore AF:CG = FH:GK = HB:KD.$ Ax. 9

EXERCISE 43

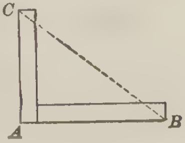
1. In the figure of § 275, suppose $AH = 5$ in., $AF = 2$ in., and $CK = 6$ in. Find the length of CG.



2. In this square PQ is $\parallel$ to AB. If a side of the square is 10 in., $DB = 14.14$ in. If $DP = 3$ in., what is the length of DQ?

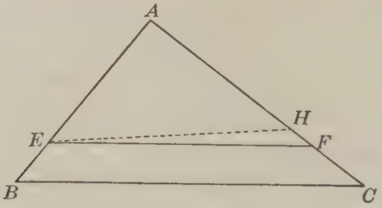
3. The sides of a triangle are respectively 3 in., 4 in., and 5 in. A line is drawn parallel to the 4-inch side, cutting the 3-inch side 1 in. from the vertex of the largest angle. Find the length of the two segments cut from the longest side.

4. Two pieces of timber 1 ft. wide are fitted together at right angles as here shown. AB is 8 ft. long, AC 6 ft. long, and the distance BC, along the dotted line, is 10 ft. A carpenter finds it necessary to saw along the dotted line. Find the length of the slanting cut across the upright piece; across the horizontal piece.



PROPOSITION X. THEOREM

276. *If a line divides two sides of a triangle proportionally from the vertex, it is parallel to the third side.*



Given the triangle ABC with EF drawn so that

$$\frac{EB}{AE}=\frac{FC}{AF}.$$

To prove that EF *is* ∥ *to* BC.

Proof. Suppose that EF is not parallel to BC.

Then from E draw some other line, as EH, parallel to BC.

Then $AB:AE=AC:AH$. § 274

(*One side of a △ is to either of its segments cut off by a line ∥ to the base as the third side is to its corresponding segment.*)

But $EB:AE=FC:AF$. Given

$\therefore EB+AE:AE=FC+AF:AF$, § 267

or $AB:AE=AC:AF$. Ax. 11

$\therefore AC:AF=AC:AH$. Ax. 8

$\therefore AF=AH$. § 263

(*For the two antecedents are equal.*)

∴ EF and EH must coincide. Post. 1

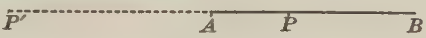
(*For their end points coincide.*)

But EH is ∥ to BC. Const.

∴ EF, which coincides with EH, is ∥ to BC. Q.E.D.

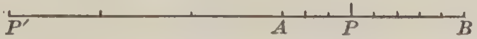
This proposition is the converse of Prop. IX.

277. Dividing a Line into Segments. If a given line AB is divided at P, a point between the extremities A and B, it is said to be *divided internally* into the segments AP and PB; and if it is divided at P', a point in the prolongation of BA, it is said to be *divided externally* into the segments AP' and $P'B$.



In either case the length of the segment is the *distance* from the point of division to an extremity of the line. If the line is divided internally, the *sum* of the segments is equal to the line; and if the line is divided externally, the *difference* of the segments is equal to the line.

Suppose it is required to divide the given line AB *internally and externally* in the same ratio; as, for example, in the ratio of the two numbers 3 and 5.



We divide AB into $3 + 5$, or 8, equal parts, and take 3 parts from A; we then have the point P, such that

$$AP : PB = 3 : 5. \quad (1)$$

Secondly, we divide AB into $5 - 3$, or 2, equal parts, and lay off on the prolongation of BA three of these equal parts; we then have the point P', such that

$$AP' : P'B = 3 : 5. \quad (2)$$

Comparing (1) and (2), we have

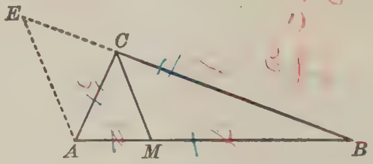
$$AP : PB = AP' : P'B.$$

278. Harmonic Division. If a given straight line is divided internally and externally into segments having the same ratio, the line is said to be *divided harmonically.*

Thus the line AB has just been divided internally and externally in the same ratio, 3 : 5, and AB is therefore said to be divided harmonically at P and P' in the ratio 3 : 5.

PROPOSITION XI. THEOREM

279. *The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.*



Given the bisector of the angle C of the triangle ABC, meeting AB at M.

To prove that $AM : MB = CA : CB.$

Proof. From A draw a line $\parallel$ to MC.

This line must meet BC produced, because BC and MC cannot both be parallel to the same line. § 94

Let this line meet BC produced at E.

Then $AM : MB = EC : CB.$ § 273

(*If a line is drawn through two sides of a △ parallel to the third side, it divides the two sides proportionally.*)

Also $\angle ACM = \angle CAE,$ § 100

(*Alt.-int. ∠s of ∥ lines are equal.*)

and $\angle MCB = \angle AEC.$ § 102

(*Ext.-int. ∠s of ∥ lines are equal.*)

But $\angle ACM = \angle MCB.$ Given

$\therefore \angle CAE = \angle AEC.$ Ax. 8

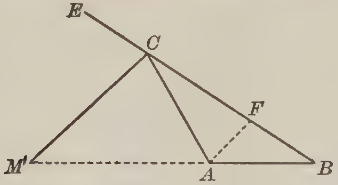
$\therefore EC = CA.$ § 76

Put CA for its equal EC in the first proportion.

Then $AM : MB = CA : CB$, by Ax. 9. Q.E.D.

PROPOSITION XII. THEOREM

280. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments which are proportional to the adjacent sides.*



Given the bisector of the exterior angle ECA of the triangle ABC, meeting BA produced at M'.

To prove that $AM' : M'B = CA : CB$.

Proof. Draw $AF \parallel$ to $M'C$, meeting BC at F.

Then $AM' : M'B = FC : CB$. § 274

(*One side of a △ is to either of its segments cut off by a line ∥ to the base as the third side is to its corresponding segment.*)

Now $\angle ECM' = \angle CFA$, § 102

and $\angle M'CA = \angle FAC$. § 100

But $\angle ECM' = \angle M'CA$. Given

$\therefore \angle CFA = \angle FAC$. Ax. 8

$\therefore CA = FC$. § 76

Put CA for its equal FC in the first proportion.

Then $AM' : M'B = CA : CB$, by Ax. 9. Q.E.D.

Discussion. In case $CA = CB$, what is the arrangement of the lines?

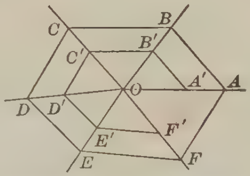
281. COROLLARY. *The bisectors of the interior angle and the exterior angle at the same vertex of a triangle, meeting the opposite side, divide that side harmonically.*

EXERCISE 44

1. In a triangle ABC, $AB = 6.5$, $CA = 6$, $BC = 7$. Find the segments of AB made by the bisector of the angle C.

2. In a triangle ABC, $CA = 7.5$, $BC = 7$, $AB = 8$. Find the segments of CA made by the bisector of the angle B.

3. The sides of a triangle are 12, 16, 20. Find the segments of the sides made by bisecting the angles.



4. If a spider, in making its web, makes $A'B' \parallel$ to AB, $B'C' \parallel$ to BC, $C'D' \parallel$ to CD, $D'E' \parallel$ to DE, and $E'F' \parallel$ to EF, and then runs a line from $F' \parallel$ to FA, will it strike the point A'? Prove it.

5. From any point O within the triangle ABC the lines OA, OB, OC are drawn and are bisected respectively by A', B', and C'. Prove that $\angle CBA = \angle C'B'A'$.

6. Prove Ex. 5 if the point O is outside the triangle.

7. From any point O within the quadrilateral $ABCD$ lines are drawn to the vertices A, B, C, D, and are bisected by A', B', C', D' Prove that $\angle CBA = \angle C'B'A'$.

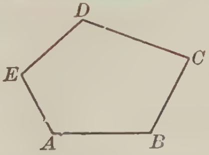
8. If a pendulum swings at the point O, cutting two parallel lines at P and Q respectively, the ratio $OP : OQ$ is constant.

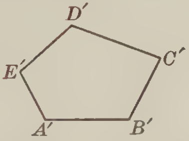
9. Through a fixed point P a line is drawn cutting a fixed line at X. PX is then divided at Y so that the ratio $PY : YX$ is constant. Find the locus of the point Y as X moves along the fixed line.

10. From the point P on the side CA of the triangle ABC parallels to the other sides are drawn meeting AB in Q and BC in R. Prove that $AQ : QB = BR : RC$.

11. In the triangle ABC, P and Q are taken on the sides CA and BC so that $AP : PC = BQ : QC$. AR is then drawn parallel to PB, meeting CB produced in R. Prove that CB is the mean proportional between CQ and CR.

282. Similar Polygons. Polygons that have their corresponding angles equal, and their corresponding sides proportional, are called *similar polygons.*





Thus the polygons $ABCDE$ and $A'B'C'D'E'$ are similar, if the ∡ A, B, C, D, E are equal respectively to the ∡ A', B', C', D', E', and if

$$AB : A'B' = BC : B'C' = CD : C'D' = DE : D'E' = EA : E'A'.$$

Similar polygons are commonly said to be of the same shape.

283. Corresponding Lines. In similar polygons those lines that are similarly situated with respect to the equal angles are called *corresponding lines.*

Corresponding lines are also called *homologous lines.*

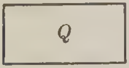
284. Ratio of Similitude. The ratio of any two corresponding lines in similar polygons is called the *ratio of similitude* of the polygons.

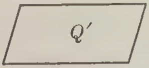
The primary idea of similarity is likeness of form. The two conditions necessary for similarity are:

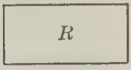
1. *For every angle in one of the figures there must be an equal angle in the other.*

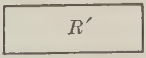
2. *The corresponding sides must be proportional.*

Thus Q and Q' are not similar; the corresponding sides are proportional, but the corresponding angles are not equal. Also R and R' are not similar; the corresponding angles are equal, but the corresponding sides are not proportional.









In the case of *triangles* either condition implies the other

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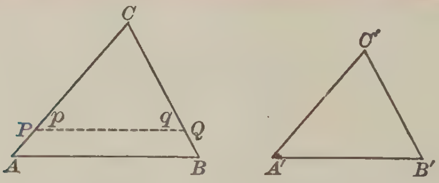
PROPOSITION XIII. THEOREM

285. *Two mutually equiangular triangles are similar.*



Given the triangles ABC and $A'B'C'$, having the angles A, B, C equal to the angles A', B', C' respectively.

To prove that the $\triangle$ *ABC and $A'B'C'$ are similar.*

Proof. Since the $\triangle$ are mutually equiangular, Given

we have only to prove that

$AB : A'B' = AC : A'C' = BC : B'C'$. § 282

Place the $\triangle A'B'C'$ on the $\triangle ABC$ so that $\angle C'$ shall coincide with its equal, the $\angle C$, and $A'B'$ take the position PQ. Post. 5

Then $\angle p = \angle A$. Given

$\therefore PQ$ is $\parallel$ to AB. § 103

$\therefore AC : PC = BC : QC$; § 274

that is, $AC : A'C' = BC : B'C'$. Ax. 9

Similarly, by placing the $\triangle A'B'C'$ on the $\triangle ABC$ so that $\angle B'$ shall coincide with its equal, the $\angle B$, we can prove that

$$AB : A'B' = BC : B'C'.$$

$\therefore AB : A'B' = AC : A'C' = BC : B'C'$. Ax. 8

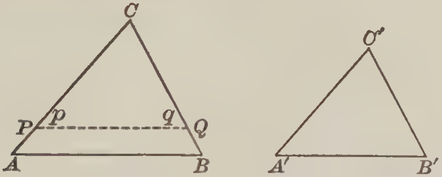
$\therefore \triangle ABC$ is similar to $\triangle A'B'C'$, by § 282. Q.E.D.

286. COROLLARY 1. *Two triangles are similar if two angles of the one are equal respectively to two angles of the other.*

287. COROLLARY 2. *Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.*

Proposition XIV. Theorem

288. *If two triangles have an angle of the one equal to an angle of the other, and the including sides proportional, they are similar.*



Given the triangles ABC and $A'B'C'$, with the angle C equal to the angle C' and with $CA : C'A' = CB : C'B'$.

To prove that the $\triangle$ *ABC and $A'B'C'$ are similar.*

Proof. Place the $\triangle A'B'C'$ on the $\triangle ABC$ so that $\angle C'$ shall coincide with its equal, the $\angle C$. Post. 5

Then let $\triangle A'B'C'$ take the position of the $\triangle PQC$.

Now $\dfrac{CA}{C'A'} = \dfrac{CB}{C'B'}$; Given

that is, $\dfrac{CA}{CP} = \dfrac{CB}{CQ}$. Ax. 9

$\therefore \dfrac{CA - CP}{CP} = \dfrac{CB - CQ}{CQ}$, § 268

or $\dfrac{PA}{CP} = \dfrac{QB}{CQ}$.

$\therefore PQ$ is $\parallel$ to AB. § 276

(*If a line divides two sides of a* $\triangle$ *proportionally, it is* $\parallel$ *to the third side.*)

$\therefore \angle p = \angle A$, and $\angle q = \angle B$. § 102

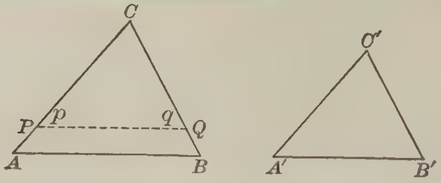
Now $\angle C = \angle C'$. Given

$\therefore \triangle PQC$ is similar to $\triangle ABC$. § 285

$\therefore \triangle A'B'C'$ is similar to $\triangle ABC$. Q. E. D.

PROPOSITION XV. THEOREM

289. *If two triangles have their sides respectively proportional, they are similar.*



Given the triangles ABC and $A'B'C'$, having

$$AB : A'B' = BC : B'C' = CA : C'A'.$$

To prove that the $\triangle$ ABC *and* $A'B'C'$ *are similar.*

Proof. Upon CA take CP equal to $C'A'$, and upon CB take CQ equal to $C'B'$; and draw PQ.

Now $CA : C'A' = BC : B'C'$. Given

Or, since $CP = C'A'$, and $CQ = C'B'$, Const.

$CA : CP = CB : CQ$. Ax. 9

Also $\angle C = \angle C$. Iden.

$\therefore$ $\triangle$ ABC and PQC are similar. § 288

(*If two* $\triangle$ *have an angle of the one equal to an angle of the other, and the including sides proportional, they are similar.*)

$\therefore CA : CP = AB : PQ$; § 282

that is, $CA : C'A' = AB : PQ$. Ax. 9

But $CA : C'A' = AB : A'B'$. Given

$\therefore AB : PQ = AB : A'B'$. Ax. 8

$\therefore PQ = A'B'$. § 263

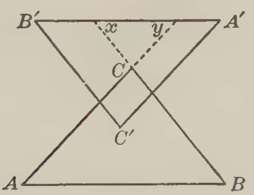
Hence the $\triangle$ PQC and $A'B'C'$ are congruent. § 80

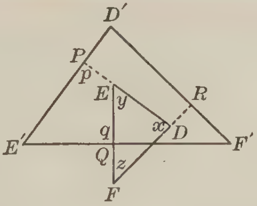
But $\triangle PQC$ has been proved similar to $\triangle ABC$.

$\therefore \triangle A'B'C'$ is similar to $\triangle ABC$. Q. E. D.

Proposition XVI. Theorem

290. *Two triangles which have their sides respectively parallel, or respectively perpendicular, are similar.*





Given the triangles ABC and $A'B'C'$, with their sides respectively parallel; and the triangles DEF and $D'E'F'$, with their sides respectively perpendicular.

To prove that 1. *the* $\triangle$s ABC *and* $A'B'C'$ *are similar;*
2. *the* $\triangle$s DEF *and* $D'E'F'$ *are similar.*

Proof. 1. Produce BC and AC to $B'A'$, forming $\angle$s x and y.

Then $\angle B = \angle x$ (§ 100), and $\angle B' = \angle x$. § 102

$\therefore \angle B = \angle B'$. Ax. 8

In like manner, $\angle A = \angle A'$.

$\therefore \triangle ABC$ is similar to $\triangle A'B'C'$. § 286

2. Produce DE and FD to meet $D'E'$ and $F'D'$ at P and R.

The quadrilateral $E'QEP$ has $\angle$s p and q right angles. Given

$\therefore \angle$s E' and PEQ are supplementary. § 144

But $\angle$s y and PEQ are supplementary. § 43

Therefore $\angle y = \angle E'$. § 58

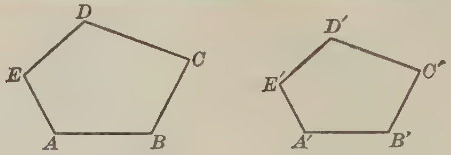
In like manner, $\angle x = \angle D'$.

$\therefore \triangle DEF$ is similar to $\triangle D'E'F'$, by § 286. Q.E.D.

Discussion. The parallel sides and the perpendicular sides respectively are corresponding sides of the triangles.

PROPOSITION XVII. THEOREM

291. *The perimeters of two similar polygons have the same ratio as any two corresponding sides.*



Given the two similar polygons $ABCDE$ and $A'B'C'D'E'$, with p and p' representing their respective perimeters.

To prove that $p:p'=AB:A'B'$.

Proof. $$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\frac{DE}{D'E'}=\frac{EA}{E'A'}. \qquad \S\ 282$$

$$\therefore \frac{AB+BC+CD+DE+EA}{A'B'+B'C'+C'D'+D'E'+E'A'}=\frac{AB}{A'B'}. \qquad \S\ 269$$

$\therefore p:p'=AB:A'B'$, by Ax. 9. Q.E.D.

EXERCISE 45

1. The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.

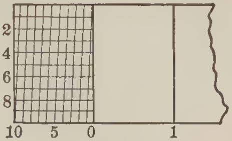
2. The base and altitude of a triangle are 15 in. and 7 in. respectively. The corresponding base of a similar triangle is 3.75 in. Find the corresponding altitude.

3. If two parallels are cut by three concurrent transversals, the corresponding segments of the parallels are proportional.

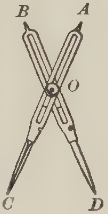
4. The point P is any point on the side OX of the angle XOY. From P a perpendicular PQ is let fall on OY. Prove that for any position of P on OX the ratio $OP:PQ$ is constant, and the ratio $PQ:OQ$ is constant.

5. In drawing a map of a triangular field with sides 75 rd., 60 rd., and 50 rd. respectively, the longest side is drawn 1 in. long. How long are the other two sides drawn?

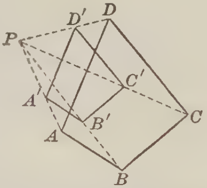
6. This figure represents part of a diagonal scale used by draftsmen. The distance from 0 to 10 is 1 centimeter, or 10 millimeters. Show how to measure 5 mm.; 1 mm.; 0.9 mm.; 0.5 mm.; 1.5 mm. On what proposition does this depend?



7. This figure represents a pair of proportional compasses used by draftsmen. By adjusting the screw at O, the lengths OA and OC, and the corresponding lengths OB and OD, may be varied proportionally. Prove that $\triangle OAB$ is always similar to $\triangle OCD$. If $OA = 3$ in. and $OC = 5$ in., then AB is what part of CD?



8. $ABCD$ is any polygon and P is any point. On AP any point A' is taken and $A'B'$ is drawn parallel to AB as shown. Then $B'C'$ and $C'D'$ are drawn parallel to BC and CD. Is $D'A'$ parallel to DA? Is $A'B'C'D'$ similar to $ABCD$? Prove it.



9. If two circles are tangent externally, the corresponding segments of two lines drawn through the point of contact and terminated by the circles are proportional.

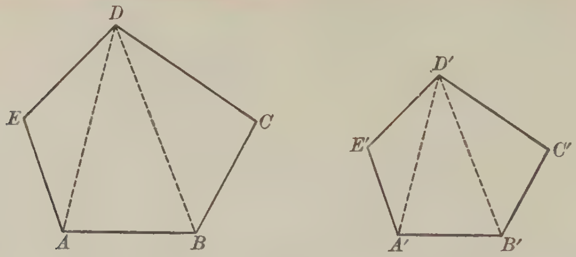
10. If two circles are tangent externally, their common external tangent is the mean proportional between their diameters.

11. AB and AC are chords drawn from any point A on a circle, and AD is a diameter. If the tangent at D intersects AB and AC at E and F, the triangles ABC and AEF are similar.

12. If AD and BE are two altitudes of the triangle ABC, the triangles DEC and ABC are similar.

PROPOSITION XVIII. THEOREM

292. *If two polygons are similar, they can be separated into the same number of triangles, similar each to each, and similarly placed.*



Given two similar polygons $ABCDE$ and $A'B'C'D'E'$ with angles A, B, C, D, E equal to angles A', B', C', D', E' respectively.

To prove that $ABCDE$ and $A'B'C'D'E'$ can be separated into the same number of triangles, similar each to each, and similarly placed.

Proof. Draw the corresponding diagonals DA, $D'A'$, and DB, $D'B'$.

Since $\angle E = \angle E'$, § 282

and $DE : D'E' = EA : E'A'$, § 282

$\therefore \triangle s\ DEA$ and $D'E'A'$ are similar. § 288

In like manner, $\triangle s\ DBC$ and $D'B'C'$ are similar.

Furthermore $\angle BAE = \angle B'A'E'$, § 282

and $\angle DAE = \angle D'A'E'$. § 282

By subtracting, $\angle BAD = \angle B'A'D'$. Ax. 2

Now $DA : D'A' = EA : E'A'$, § 282

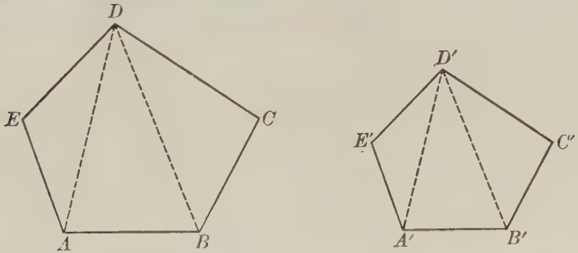
and $AB : A'B' = EA : E'A'$. § 282

$\therefore DA : D'A' = AB : A'B'$. Ax. 8

$\therefore \triangle s\ DAB$ and $D'A'B'$ are similar, by § 288. Q.E.D.

PROPOSITION XIX. THEOREM

293. *If two polygons are composed of the same number of triangles, similar each to each, and similarly placed, the polygons are similar.*



Given two polygons $ABCDE$ and $A'B'C'D'E'$ composed of the triangles DEA, DAB, DBC, similar respectively to the triangles $D'E'A'$, $D'A'B'$, $D'B'C'$, and similarly placed.

To prove that $ABCDE$ is similar to $A'B'C'D'E'$.

Proof. $\angle E = \angle E'$. § 282

Also $\angle DAE = \angle D'A'E'$,

and $\angle BAD = \angle B'A'D'$. § 282

By adding, $\angle BAE = \angle B'A'E'$. Ax. 1

Similarly $\angle CBA = \angle C'B'A'$, and $\angle EDC = \angle E'D'C'$.

Again, $\angle C = \angle C'$. § 282

Hence the polygons are mutually equiangular.

Also $\dfrac{DE}{D'E'} = \dfrac{EA}{E'A'} = \dfrac{DA}{D'A'} = \dfrac{AB}{A'B'} = \dfrac{DB}{D'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$. § 282

Hence the polygons are not only mutually equiangular but they have their corresponding sides proportional.

Therefore the polygons are similar, by § 282. Q.E.D

This proposition is the converse of Prop. XVIII.

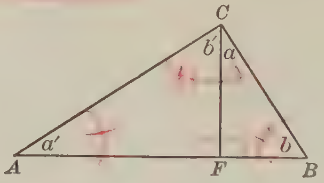
PROPOSITION XX. THEOREM

294. *If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse:*

1. *The triangles thus formed are similar to the given triangle, and are similar to each other.*

2. *The perpendicular is the mean proportional between the segments of the hypotenuse.*

3. *Each of the other sides is the mean proportional between the hypotenuse and the segment of the hypotenuse adjacent to that side.*



Given the right triangle *ABC*, with *CF* drawn from the vertex of the right angle *C*, perpendicular to *AB*.

1. *To prove that the* ⟁ *BCA*, *CFA*, *BFC* *are similar.*

Proof. Since the $\angle a'$ is common to the rt. ⟁ *CFA* and *BCA*,

$\therefore$ these ⟁ are similar. § 287

Since the $\angle b$ is common to the rt. ⟁ *BFC* and *BCA*,

$\therefore$ these ⟁ are similar. § 287

Since the ⟁ *CFA* and *BFC* are each similar to $\triangle$ *BCA*,

$\therefore$ these ⟁ are mutually equiangular. § 282

Therefore the ⟁ *CFA* and *BFC* are similar, by § 285. Q. E. D.

2. *To prove that* $AF : CF = CF : FB$.

Proof. In the similar ⟁ *CFA* and *BFC*,

$AF : CF = CF : FB$, by § 282. Q. E. D.

3. *To prove that* $AB : AC = AC : AF,$
and $AB : BC = BC : BF.$

Proof. In the similar $\triangle$ BCA and CFA,

$$AB : AC = AC : AF. \qquad \text{§ 282}$$

In the similar $\triangle$ BCA and BFC,

$AB : BC = BC : BF$, by § 282. Q.E.D.

295. COROLLARY 1. *The squares on the two sides of a right triangle are proportional to the segments of the hypotenuse adjacent to those sides.*

From the proportions in § 294, 3,

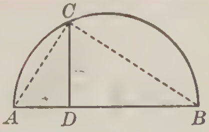
$$\overline{AC}^2 = AB \times AF, \text{ and } \overline{BC}^2 = AB \times BF. \qquad \text{§ 261}$$

Hence $$\frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AB \times AF}{AB \times BF} = \frac{AF}{BF}. \qquad \text{Ax. 4}$$

296. COROLLARY 2. *The square on the hypotenuse and the square on either side of a right triangle are proportional to the hypotenuse and the segment of the hypotenuse adjacent to that side.*

Since $$\overline{AB}^2 = AB \times AB, \qquad \text{Iden.}$$
and, as in § 295, $$\overline{AC}^2 = AB \times AF, \qquad \text{§ 261}$$
$$\therefore \frac{\overline{AB}^2}{\overline{AC}^2} = \frac{AB \times AB}{AB \times AF} = \frac{AB}{AF}. \qquad \text{Ax. 4}$$

297. COROLLARY 3. *The perpendicular from any point on a circle to a diameter is the mean proportional between the segments of the diameter.*



298. COROLLARY 4. *If a perpendicular is drawn from any point on a circle to a diameter, the chord from that point to either extremity of the diameter is the mean proportional between the diameter and the segment adjacent to that chord.*

EXERCISE 46

1. The perimeters of two similar polygons are 18 in. and 14 in. If a side of the first is 3 in., find the corresponding side of the second.

2. In two similar triangles, ABC and $A'B'C'$, $AB = 6$ in., $BC = 7$ in., $CA = 8$ in., and $A'B' = 9$ in. Find $B'C'$ and $C'A'$.

3. The corresponding bases of two similar triangles are 11 in. and 13 in. The altitude of the first is 6 in. Find the corresponding altitude of the second.

4. The perimeter of an equilateral triangle is 51 in. Find the side of an equilateral triangle of half the altitude.

5. The sides of a polygon are 2 in., $2\frac{1}{2}$ in., $3\frac{1}{4}$ in., 3 in., and 5 in. Find the perimeter of a similar polygon whose longest side is 7 in.

6. The perimeter of an isosceles triangle is 13, and the ratio of one of the equal sides to the base is $1\frac{2}{3}$. Find the three sides.

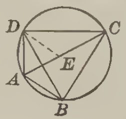
7. The perimeter of a rectangle is 48 in., and the ratio of two of the sides is $\frac{5}{7}$. Find the sides.

8. In drawing a map to the scale $\frac{1}{1000000}$, what length will represent the sides of a county that is a rectangle 25 mi. long and 10 mi. wide? Answer to the nearest tenth of an inch.

9. Two circles touch at P. Through P three lines are drawn, meeting one circle in A, B, C, and the other in A', B', C' respectively. Prove the triangles ABC, $A'B'C'$ similar.

10. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the smaller circle.

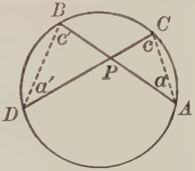
11. In an inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides.



Draw DE, making $\angle EDC = \angle ADB$. The $\triangle$s ABD and ECD are similar; and the $\triangle$s BCD and AED are similar.

PROPOSITION XXI. THEOREM

299. *If two chords intersect within a circle, the product of the segments of the one is equal to the product of the segments of the other.*



Given the chords AB and CD, intersecting at P.

To prove that $PA \times PB = PC \times PD$.

Proof. Draw AC and BD.

Then since $\angle a = \angle a'$, § 214

(*Each is measured by $\frac{1}{2}$ arc CB.*)

and $\angle c = \angle c'$, § 214

(*Each is measured by $\frac{1}{2}$ arc DA.*)

$\therefore$ the $\triangle$ CPA and BPD are similar. § 286

$\therefore PA : PD = PC : PB$. § 282

$\therefore PA \times PB = PC \times PD$, by § 261. Q.E.D.

300. COROLLARY. *If two chords intersect within a circle, the segments of the one are reciprocally proportional to the segments of the other.*

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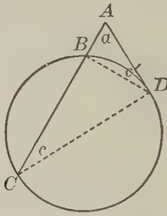
This means, for example, that $PA : PD$ equals the *reciprocal* of $PB : PC$, or equals $PC : PB$, as shown above.

301. Secant to a Circle. *A secant from an external point to a circle* is understood to mean the segment of the secant that lies between the given external point and the *second point* of intersection of the secant and circle.

PROPOSITION XXII. THEOREM

302. *If from a point outside a circle a secant and a tangent are drawn, the tangent is the mean proportional between the secant and its external segment.*



Given a tangent AD and a secant AC drawn from the point A to the circle BCD.

To prove that $AC : AD = AD : AB$.

Proof. Draw DC and DB.

Now $\angle c$ is measured by $\frac{1}{2}$ arc DB, § 214

and $\angle c'$ is measured by $\frac{1}{2}$ arc DB. § 220

$\therefore \angle c = \angle c'$.

Then in the $\triangle$ ADC and ABD,

$\angle a = \angle a$, Iden.

and $\angle c = \angle c'$.

$\therefore \triangle$ ADC and ABD are similar. § 286

$\therefore AC : AD = AD : AB$, by § 282. Q.E.D.

303. COROLLARY. *If from a fixed point outside a circle a secant is drawn, the product of the secant and its external segment is constant in whatever direction the secant is drawn.*

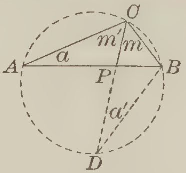
Since $AC : AD = AD : AB$, § 302

$\therefore AC \times AB = \overline{AD}^2$. § 261

Since AD is constant (§ 192), therefore $AC \times AB$ is constant.

PROPOSITION XXIII. THEOREM

304. *The square on the bisector of an angle of a triangle is equal to the product of the sides of this angle diminished by the product of the segments made by the bisector upon the third side of the triangle.*



Given the line *CP* bisecting the angle *ACB* of the triangle *ABC*.

To prove that $\overline{CP}^2 = CA \times BC - AP \times PB$.

Proof. Circumscribe the ⊙ *BCA* about the △ *ABC*. § 240

Produce *CP* to meet the circle in *D*, and draw *BD*.

Then in the △s *BCD* and *PCA*,

$\angle m = \angle m'$, Given

and $\angle a' = \angle a$. § 214

(*Each is measured by ½ arc BC.*)

∴ the △s *BCD* and *PCA* are similar. § 286

$\therefore CD : CA = BC : CP$. § 282

$\therefore CA \times BC = CD \times CP$ § 261

$= (CP + PD)\, CP$ Ax. 9

$= \overline{CP}^2 + CP \times PD$.

But $CP \times PD = AP \times PB$. § 299

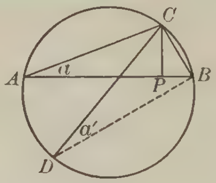
$\therefore CA \times BC = \overline{CP}^2 + AP \times PB$. Ax. 9

$\therefore \overline{CP}^2 = CA \times BC - AP \times PB$, by Ax. 2. Q.E.D.

This theorem enables us to compute the bisectors of the angles of a triangle terminated by the opposite sides, if the sides are known. The theorem may be omitted without destroying the sequence.

PROPOSITION XXIV. THEOREM

305. *In any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle by the altitude upon the third side.*



Given the triangle *ABC* with *CP* the altitude, *ADBC* the circle circumscribed about the triangle *ABC*, and *CD* a diameter.

To prove that $CA \times BC = CD \times CP$.

Proof. Draw BD.

Then in the $\triangle$ APC and DBC,

$\angle CPA$ is a rt. $\angle$, Given

$\angle CBD$ is a rt. $\angle$, § 215

$\angle a$ is measured by $\frac{1}{2}$ arc BC,

and $\angle a'$ is measured by $\frac{1}{2}$ arc BC. § 214

$\therefore \angle a = \angle a'$.

$\therefore$ $\triangle$ APC and DBC are similar. § 287

(*Two rt. $\triangle$ are similar if an acute $\angle$ of the one is equal to an acute $\angle$ of the other.*)

$\therefore CA : CD = CP : BC$. § 282

$\therefore CA \times BC = CD \times CP$, by § 261. Q.E.D.

This theorem may be omitted without destroying the sequence. Props. XXIII and XXIV are occasionally demanded in college entrance examinations, but they are not necessary for proving subsequent propositions or for any of the exercises. Teachers may therefore use their judgment as to including them.

EXERCISE 47

1. The tangents to two intersecting circles, drawn from any point in their common chord produced, are equal.

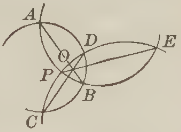
2. The common chord of two intersecting circles, if produced, bisects their common tangents.

3. If two circles are tangent externally, the common internal tangent bisects the two common external tangents.

4. If a line drawn from a vertex of a triangle divides the opposite side into segments proportional to the adjacent sides, the line bisects the angle at the vertex.

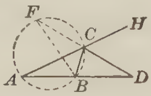
5. If three circles intersect one another, the common chords are concurrent.

Let two of the chords, AB and CD, meet at O. Join the point of intersection E to O, and suppose that EO produced meets the same two circles at two different points P and Q. Then prove that $OP = OQ$ (§ 299), and hence that the points P and Q coincide.



6. The square on the bisector of an exterior angle of a triangle is equal to the product of the segments determined by this bisector upon the opposite side, diminished by the product of the other two sides.

Let CD bisect the exterior ∠BCH of the △ABC. △s ADC and FBC are similar (§ 286). Apply § 303.



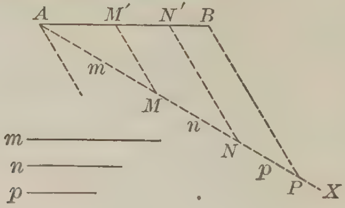
7. If the line of centers of two circles meets the circles at the consecutive points A, B, C, D, and meets the common external tangent at P, then $PA \times PD = PB \times PC$.

8. The line of centers of two circles meets the common external tangent at P, and a secant is drawn from P, cutting the circles at the consecutive points E, F, G, H. Prove that $PE \times PH = PF \times PG$.

Draw radii to the points of contact, and to E, F, G, H. Let fall ⊥s on PH from the centers of the ⓢ. The various pairs of △s are similar.

PROPOSITION XXV. PROBLEM

306. *To divide a given line into parts proportional to any number of given lines.*



Given the lines AB, m, n, and p.

Required to divide AB into parts proportional to m, n, and p.

Construction. Draw AX, making any convenient ∠ with AB.

On AX take AM equal to m,

MN equal to n, and NP equal to p.

Draw BP.

From N draw NN' ∥ to PB,

and from M draw MM' ∥ to PB. § 233

Then M' and N' are the division points required. Q.E.F.

Proof. Through A draw a line ∥ to PB. § 233

$$\frac{AM'}{AM}=\frac{M'N'}{MN}=\frac{N'B}{NP}. \qquad \S\ 275$$

(*Three or more ∥ lines cut off proportional intercepts on any two transversals.*)

Substituting m, n, and p for their equals AM, MN, and NP,

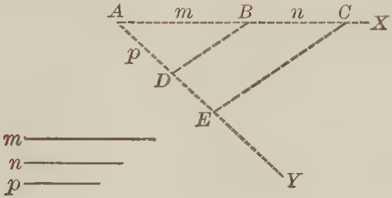
we have
$$\frac{AM'}{m}=\frac{M'N'}{n}=\frac{N'B}{p}. \qquad \text{Ax. 9}$$

This means that AB has been divided as required. Q.E.D.

In like manner, we may divide AB into parts proportional to any number of given lines.

Proposition XXVI. Problem

307. *To find the fourth proportional to three given lines.*



Given the three lines *m*, *n*, and *p*.

Required to find the fourth proportional to m, n, and p.

Construction. Draw two lines AX and AY containing any convenient angle.

On AX take AB equal to m,

and take BC equal to n.

On AY take AD equal to p.

Draw BD.

From C draw $CE \parallel$ to BD, meeting AY at E. § 233

Then DE is the fourth proportional required. Q.E.F.

Proof. $AB : BC = AD : DE.$ § 273

(*If a line is drawn through two sides of a △ ∥ to the third side, it divides the two sides proportionally.*)

Substituting m, n, and p for their equals AB, BC, and AD,

we have $m : n = p : DE.$ Ax. 9

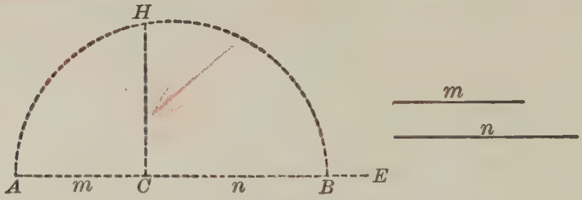
Therefore DE is the fourth proportional to m, n, and p, by § 259. Q.E.D.

308. Corollary. *To find the third proportional to two given lines.*

In the above proof take m, n, n as the given lines instead of m, n, p.

Proposition XXVII. Problem

309. *To find the mean proportional between two given lines.*



Given the two lines m and n.

Required to find the mean proportional between m and n.

Construction. Draw any line AE, and on AE take AC equal to m, and CB equal to n.

On AB as a diameter describe a semicircle.

At C erect the ⊥ CH, meeting the circle at H. § 228

Then CH is the mean proportional between m and n. Q.E.F.

Proof. $AC : CH = CH : CB$. § 297

(*The ⊥ from any point on a circle to a diameter is the mean proportional between the segments of the diameter.*)

Substituting for AC and CB their equals m and n,

we have $m : CH = CH : n$, by Ax. 9. Q.E.D.

310. Extreme and Mean Ratio. If a line is divided into two segments such that one segment is the mean proportional between the whole line and the other segment, the line is said to be divided in *extreme and mean ratio.*

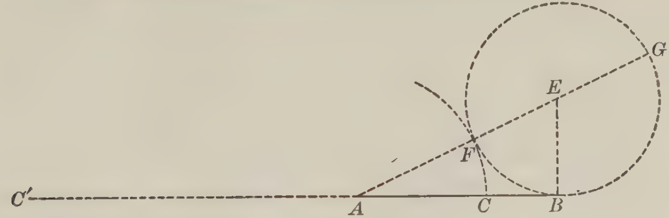
E.g. the line a is divided in extreme and mean ratio, if a segment x is found such that

$$a : x = x : a - x.$$

The division of a line in extreme and mean ratio is often called the *Golden Section.*

Proposition XXVIII. Problem

311. *To divide a given line in extreme and mean ratio.*



Given the line AB.

Required to divide AB *in extreme and mean ratio.*

Construction. At B erect a ⊥ BE equal to half of AB. § 228

From E as a center, with a radius equal to EB, describe a ⊙.

Draw AE, meeting the circle at F and G.

On AB take AC equal to AF.

On BA produced take AC' equal to AG.

Then AB is divided internally at C and externally at C' in extreme and mean ratio.

That is, $AB : AC = AC : CB$, and $AB : AC' = AC' : C'B$. Q.E.F.

Proof. $AG : AB = AB : AF$. § 302

From $AG : AB = AB : AF$,

$$AG - AB : AB = AB - AF : AF.$$

$$\therefore AG - FG : AB = AB - AC : AC.$$

$$\therefore AC : AB = CB : AC.$$

$$\therefore AB : AC = AC : CB,$$

by inversion, § 266. Q.E.D.

$\therefore AB : AG = AF : AB$. § 266

$$\therefore AB + AG : AG = AF + AB : AB$$

$$\therefore AB + AC' : AC' = AF + FG : AB$$

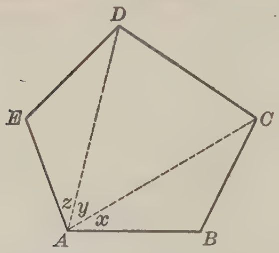
$$\therefore C'B : AC' = AC' : AB.$$

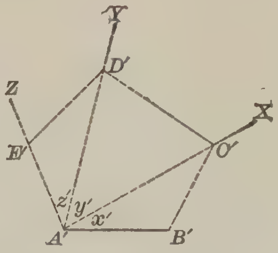
$$\therefore AB : AC' = AC' : C'B,$$

by §§ 261, 264. Q.E.D

Proposition XXIX. Problem

312. *Upon a given line corresponding to a given side of a given polygon, to construct a polygon similar to the given polygon.*





Given the line $A'B'$ and the polygon $ABCDE$.

Required to construct on $A'B'$, corresponding to AB, a polygon similar to the polygon $ABCDE$.

Construction. From A draw the diagonals AD and AC.

From A' draw $A'X$, $A'Y$, and $A'Z$, making $\angle$s x', y', and z' equal respectively to $\angle$s x, y, and z. § 232

From B' draw a line, making $\angle B'$ equal to $\angle B$,
and meeting $A'X$ at C'.

From C' draw a line, making $\angle D'C'B'$ equal to $\angle DCB$,
and meeting $A'Y$ at D'.

From D' draw a line, making $\angle E'D'C'$ equal to $\angle EDC$,
and meeting $A'Z$ at E'.

Then $A'B'C'D'E'$ is the required polygon. Q.E.F.

Proof. The $\triangle$s ABC and $A'B'C'$, the $\triangle$s ACD and $A'C'D'$, and the $\triangle$s ADE and $A'D'E'$, are similar. § 286

Therefore the two polygons are similar, by § 293. Q.E.D.

EXERCISE 48

1. If a and b are two given lines, construct a line equal to x, where $x = \sqrt{ab}$. Consider the special case of $a = 2$, $b = 3$.

2. If m and n are two given lines, construct a line equal to x, where $x = \sqrt{2mn}$.

3. Determine both by geometric construction and arithmetically the third proportional to the lines $1\frac{1}{2}$ in. and 2 in.

4. Determine both by geometric construction and arithmetically the third proportional to the lines 4 in. and 3 in.

5. Determine both by geometric construction and arithmetically the fourth proportional to the lines $1\frac{1}{4}$ in., 2 in., and $2\frac{1}{4}$ in.

6. Determine both by geometric construction and arithmetically the mean proportional between the lines 1.2 in. and 2.7 in.

7. Find geometrically the square root of 5. Measure the line and thus determine the approximate arithmetical value.

8. A map is drawn to the scale of 1 in. to 50 mi. How far apart are two places that are $2\frac{3}{32}$ in. apart on the map?

9. Find by geometric construction and arithmetically the third proportional to the two lines $1\frac{3}{16}$ in. and $2\frac{5}{8}$ in.

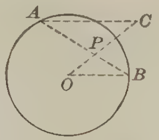
10. Divide a line 1 in. long in extreme and mean ratio. Measure the two segments and determine their lengths to the nearest sixteenth of an inch.

11. Divide a line 5 in. long in extreme and mean ratio. Measure the two segments and determine their lengths to the nearest sixteenth of an inch.

12. Divide a line 6 in. long in extreme and mean ratio. Measure the two segments and determine their lengths to the nearest sixteenth of an inch.

The propositions on this page are taken from recent college entrance examination papers.

13. Through a given point P within a given circle to draw a chord AB so that the ratio $AP : BP$ shall equal a given ratio $m : n$.

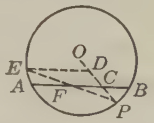


Draw OPC so that $OP : PC = n : m$.

Draw CA equal to the fourth proportional to n, m, and the radius of the circle.

14. To draw two lines making an angle of 60°, and to construct all the circles of $\frac{1}{2}$ in. radius that are tangent to both lines.

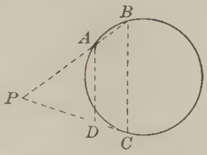
15. To draw through a given point P in the arc subtended by a chord AB a chord which shall be bisected by AB.



On radius OP take CD equal to CP. Draw $DE \parallel$ to BA.

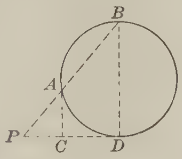
16. To construct two circles of radii $\frac{1}{2}$ in. and 1 in. respectively, which shall be tangent externally, and to construct a third circle of radius 3 in., which shall be tangent to each of these two circles and inclose both of them.

17. To draw through a given external point P a secant PAB to a given circle so that the ratio $PA : AB$ shall equal the given ratio $m : n$.



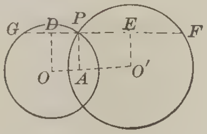
Draw the tangent PC. Make $PD : DC = m : n$. $PA : PC = PC : PB$.

18. To draw through a given external point P a secant PAB to a given circle so that $\overline{AB}^2 = PA \times PB$.



19. An equilateral triangle ABC is 2 in. on a side. To construct a circle which shall be tangent to AB at the point A and shall pass through the point C.

20. To draw through one of the points of intersection of two circles a secant so that the two chords that are formed shall be in the given ratio $m : n$

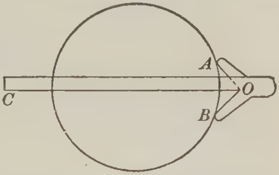


21. In a circle of 3 in. radius chords are drawn through a point 1 in. from the center. What is the product of the segments of these chords?

22. The chord AB is 3 in. long, and it is produced through B to the point P so that PB is equal to 12 in. Find the tangent from P.

23. Two lines AB and CD intersect at O. How would you ascertain, by measuring OA, OB, OC, and OD, whether or not the four points A, B, C, and D lie on the same circle?

24. This figure represents an instrument for finding the centers of circular plates or sections of shafts. OC is a ruler that bisects the angle AOB, and AO and OB are equal. Show that, if A and B rest on the circle, OC passes through the center, and that by drawing two lines the center can be found.



25. If three circles are tangent externally each to the other two, the tangents at their points of contact pass through the center of the circle inscribed in the triangle formed by joining the centers of the three given circles.

26. In the isosceles triangle ABC, C is a right angle, and AC is 4 in. With A as center and a radius 2 in. a circle is described. Required to describe another circle tangent to the first and also tangent to BC at the point B.

27. Find the center of a circle of $\frac{1}{2}$ in. radius, so drawn in a semicircle of radius 2 in. as to be tangent to the semicircle itself and to its diameter.

28. To inscribe in a given circle a triangle similar to a given triangle.

29. To draw two straight line-segments, having given their sum and their ratio.

EXERCISE 49

Review Questions

1. What is meant by ratio? by proportion?

2. If $a : b = c : d$, write four other proportions involving these quantities.

3. If $a : b = c : d$, is it true in general that $a + 1 : b + 1 = c : d$? Is it ever true?

4. When is a line divided harmonically? The bisectors of what angles of a triangle divide the opposite side harmonically?

5. What are the two conditions necessary for the similarity of two polygons?

6. Are two mutually equiangular triangles similar? Are two mutually equiangular polygons always similar?

7. Are two triangles similar if their corresponding sides are proportional? Are two polygons always similar if their corresponding sides are proportional?

8. If two triangles have their sides respectively parallel, are they similar? Is this true of polygons in general?

9. If two triangles have their sides respectively perpendicular, are they similar? Is this true of polygons in general?

10. Complete in two ways: The perimeters of two similar polygons have the same ratio as any two corresponding ⋯.

11. If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse, state three geometric truths that follow.

12. If two secants intersect outside, on, or within a circle, what geometric truth follows?

13. How would you proceed to divide a straight line into seven equal parts?

14. How would you proceed to find the square root of 7 by measuring the length of a line?

BOOK IV

AREAS OF POLYGONS

313. Unit of Surface. A square the side of which is a unit of length is called a *unit of surface.*

Thus a square that is 1 inch long is 1 square inch, and a square that is 1 mile long is 1 square mile. If we are measuring the dimensions of a room in feet, we measure the surface of the floor in square feet. In the same way we may measure the page of this book in square inches and the area of a state in square miles.

314. Area of a Surface. The measure of a surface, expressed in units of surface, is called its *area*.

If a room is 20 feet long and 15 feet wide, the floor contains 300 square feet. Therefore the area of the floor is 300 square feet. Usually the two sides of a rectangle are not commensurable, although by means of fractions we may measure them to any required degree of approximation. The incommensurable cases in theorems like Prop. I of this Book may be omitted without interfering with the sequence of the course.

315. Equivalent Figures. Plane figures that have equal areas are said to be *equivalent.*

In propositions relating to areas the words *rectangle, triangle*, etc., are often used for *area of rectangle, area of triangle*, etc.

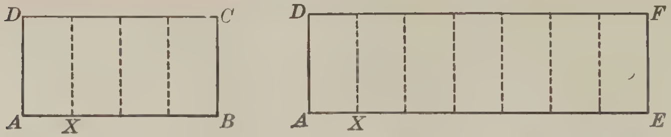
Since congruent figures may be made to coincide, congruent figures are manifestly equivalent.

Because their areas are equal, equivalent figures are frequently spoken of as equal figures. The symbol $=$ is used both for "equivalent" and for "congruent," the sense determining which meaning is to be assigned to it. Occasionally these symbols are used: $\hat{=}$, $\cong$, or $\equiv$ for congruent, $=$ for equal, and $\Bumpeq$ for equivalent.

Since the word *congruent* means "identically equal," the word *equal* is often used to mean "equivalent."

PROPOSITION I. THEOREM

316. *Two rectangles having equal altitudes are to each other as their bases.*



Given the rectangles AC and AF, having equal altitudes AD.

To prove that $\square AC : \square AF = \text{base } AB : \text{base } AE.$

CASE 1. *When AB and AE are commensurable.*

Proof. Suppose AB and AE have a common measure, as AX. Suppose AX is contained m times in AB and n times in AE.

Then $AB : AE = m : n.$

(*For m and n are the numerical measures of AB and AE.*)

Apply AX as a unit of measure to AB and AE, and at the several points of division erect $\perp$s.

These $\perp$s are all $\perp$ to the upper bases, § 97

and these $\perp$s are all equal. § 128

Since to each base equal to AX there is one rectangle,

$\therefore \square AC$ is divided into m rectangles,

and $\square AF$ is divided into n rectangles. § 119

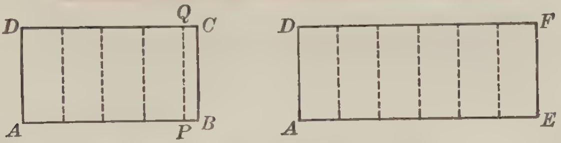
These rectangles are all congruent. § 133

$\therefore \square AC : \square AF = m : n.$

$\therefore \square AC : \square AF = AB : AE$, by Ax. 8. Q.E.D.

In this proposition we again meet the incommensurable case, as on pages 116 and 157. This case is considered on page 193 and may be omitted without destroying the sequence of the propositions.

Case 2. *When AB and AE are incommensurable.*



Proof. Divide AE into any number of equal parts, and apply one of these parts to AB as many times as AB will contain it.

Since AB and AE are incommensurable, a certain number of these parts will extend from A to some point P, leaving a remainder PB less than one of them. Draw $PQ \perp$ to AB.

Then $$\frac{\square AQ}{\square AF} = \frac{AP}{AE}.$$ Case 1

By increasing the *number* of equal parts into which AE is divided we can diminish the *length* of each, and therefore can make PB less than any assigned positive value, however small.

Hence PB approaches zero as a limit, as the number of parts of AE is indefinitely increased, and at the same time the corresponding $\square PC$ approaches zero as a limit. § 204

Therefore AP approaches AB as a limit, and $\square AQ$ approaches $\square AC$ as a limit.

$$\therefore \text{ the variable } \frac{AP}{AE} \text{ approaches } \frac{AB}{AE} \text{ as a limit,}$$

$$\text{and the variable } \frac{\square AQ}{\square AF} \text{ approaches } \frac{\square AC}{\square AF} \text{ as a limit.}$$

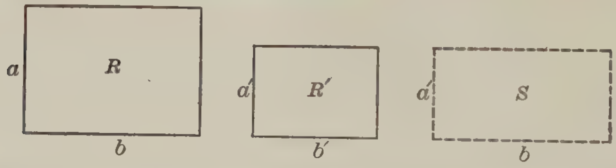
But $\frac{AP}{AE}$ is always equal to $\frac{\square AQ}{\square AF}$, as AP varies in value and approaches AB as a limit. Case 1

$$\therefore \frac{\square AC}{\square AF} = \frac{AB}{AE}, \text{ by § 207.}$$ Q.E.D.

317. Corollary. *Two rectangles having equal bases are to each other as their altitudes.*

PROPOSITION II. THEOREM

318. *Two rectangles are to each other as the products of their bases by their altitudes.*



Given the rectangles R and R', having for the numerical measure of their bases b and b', and of their altitudes a and a' respectively.

To prove that $\dfrac{R}{R'} = \dfrac{ab}{a'b'}$.

Proof. Construct the rectangle S, with its base equal to that of R, and its altitude equal to that of R'.

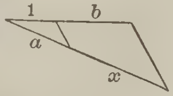
Then $\dfrac{R}{S} = \dfrac{a}{a'}$, § 317

and $\dfrac{S}{R'} = \dfrac{b}{b'}$. § 316

Since we are considering areas, we may treat R, R', and S as numbers and take the products of the corresponding members of these equations. § 272

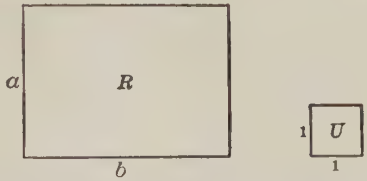
We therefore have $\dfrac{R}{R'} = \dfrac{ab}{a'b'}$, by Ax. 3. Q. E. D.

319. Products of Lines. When we speak of the product of a and b we mean the product of their numerical values. It is possible, however, to think of a line as the product of two lines, by changing the definition of multiplication. Thus in this figure in which two parallels are cut by two intersecting transversals, we have $1 : a = b : x$. Therefore $x = ab$. In the same way we may find xc, or abc, the product of three lines.



PROPOSITION III. THEOREM

320. *The area of a rectangle is equal to the product of its base by its altitude.*



Given the rectangle R, having for the numerical measure of its base and altitude b and a respectively.

To prove that the area of $R = ab$.

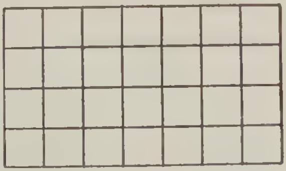
Proof. Let U be the unit of surface. § 313

Then $$\frac{R}{U} = \frac{ab}{1 \times 1} = ab.$$ § 318

But $\frac{R}{U} =$ the *number* of units of surface in R, i.e. the area of R. § 314

$\therefore$ the area of $R = ab$, by Ax. 8. Q.E.D.

321. Practical Measures. When the base and altitude both contain the linear unit an integral number of times, this proposition is rendered evident by dividing the rectangle into squares, each equal to the unit of surface.



Thus, if the base contains seven linear units and the altitude four, the rectangle may be divided into twenty-eight squares, each equal to the unit of surface. Practically this is the way in which we conceive the measure of all rectangles. Even if the sides are incommensurable, we cannot determine this by any measuring instrument. If they seem to be incommensurable with a unit of a thousandth of an inch, they might not seem to be incommensurable with a unit of a millionth of an inch.

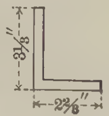
EXERCISE 50

1. A square and a rectangle have equal perimeters, 144 yd., and the length of the rectangle is five times the breadth. Compare the areas of the square and rectangle.

2. On a certain map the linear scale is 1 in. to 10 mi. How many acres are represented by a square $\frac{3}{4}$ in. on a side?

3. Find the ratio of a lot 90 ft. long by 60 ft. wide to a field 40 rd. long by 20 rd. wide.

4. Find the area of a gravel walk 3 ft. 6 in. wide, which surrounds a rectangular plot of grass 40 ft. long and 25 ft. wide. Make a drawing to scale before beginning to compute.



5. Find the number of square inches in this cross section of an L beam, the thickness being $\frac{1}{3}$ in.

6. What is the perimeter of a square field that contains exactly an acre?

7. A machine for planing iron plates planes a space $\frac{1}{2}$ in. wide and 18 ft. long in 1 min. How long will it take to plane a plate 22 ft. 6 in. long and 4 ft. 6 in. wide, allowing 51 min. for adjusting the machine?

8. How many tiles, each 8 in. square, will it take to cover a floor 24 ft. 8 in. long by 16 ft. wide?

9. A rectangle having an area of 48 sq. in. is three times as long as wide. What are the dimensions?

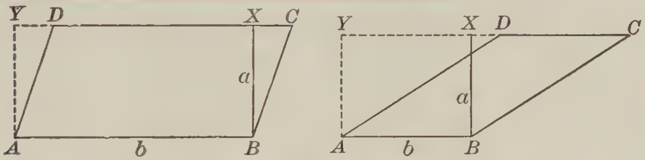
10. The length of a rectangle is four times the width. If the perimeter is 60 ft., what is the area?

11. From two adjacent sides of a rectangular field 60 rd. long and 40 rd. wide a road is cut 4 rd. wide. How many acres are cut off for the road?

12. From one end of a rectangular sheet of iron 10 in. long a square piece is cut off leaving 25 sq. in. in the rest of the sheet. How wide is the sheet?

PROPOSITION IV. THEOREM

322. *The area of a parallelogram is equal to the product of its base by its altitude.*



Given the parallelogram $ABCD$, with base b and altitude a.

To prove that the area of the ▱ $ABCD = ab$.

Proof. From B draw $BX \perp$ to CD or to CD produced, and from A draw $AY \perp$ to CD produced.

Then $ABXY$ is a rectangle, with base b and altitude a.

Since $AY = BX$, and $AD = BC$, § 125

∴ the rt. △s ADY and BCX are congruent. § 89

From $ABCY$ take the $\triangle BCX$; the □ $ABXY$ is left.

From $ABCY$ take the $\triangle ADY$; the ▱ $ABCD$ is left.

∴ □ $ABXY$ = ▱ $ABCD$. Ax. 2

But the area of the □ $ABXY = ab$. § 320

∴ the area of the ▱ $ABCD = ab$, by Ax. 8. Q.E.D.

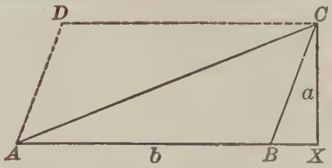
323. COROLLARY 1. *Parallelograms having equal bases and equal altitudes are equivalent.*

324. COROLLARY 2. *Parallelograms having equal bases are to each other as their altitudes; parallelograms having equal altitudes are to each other as their bases; any two parallelograms are to each other as the products of their bases by their altitudes.*

This was regarded as very interesting by the ancients, since an ignorant person might think it impossible that the areas of two parallelograms could remain the same although their perimeters differed without limit

Proposition V. Theorem

325. *The area of a triangle is equal to half the product of its base by its altitude.*



A = [illegible]

Given the triangle *ABC*, with altitude *a* and base *b*.

To prove that the area of the $\triangle ABC = \frac{1}{2}ab$.

Proof. With AB and BC as adjacent sides construct the parallelogram $ABCD$. § 238

Then $\triangle ABC = \frac{1}{2} \square ABCD$. § 126

But the area of the $\square ABCD = ab$. § 322

$\therefore$ the area of the $\triangle ABC = \frac{1}{2}ab$, by Ax. 4. Q.E.D.

326. Corollary 1. *Triangles having equal bases and equal altitudes are equivalent.*

327. Corollary 2. *Triangles having equal bases are to each other as their altitudes; triangles having equal altitudes are to each other as their bases; any two triangles are to each other as the products of their bases by their altitudes.*

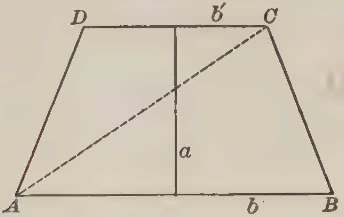
Has this been proved for rectangles? What is the relation of a triangle to a rectangle of equal base and equal altitude? What must then be the relations of triangles to one another?

328. Corollary 3. *The product of the sides of a right triangle is equal to the product of the hypotenuse by the altitude from the vertex of the right angle.*

How is the area of a right triangle found in terms of the sides of the right angle? in terms of the hypotenuse and altitude? How do these results compare?

PROPOSITION VI. THEOREM

329. *The area of a trapezoid is equal to half the product of the sum of its bases by its altitude.*



Given the trapezoid *ABCD*, with bases *b* and *b′* and altitude *a*.

To prove that the area of $ABCD = \frac{1}{2}a(b + b')$.

Proof. Draw the diagonal AC.

Then the area of the $\triangle ABC = \frac{1}{2}ab$,

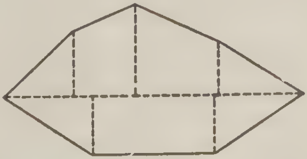
and the area of the $\triangle ACD = \frac{1}{2}ab'$. § 325

$\therefore$ the area of $ABCD = \frac{1}{2}a(b + b')$, by Ax. 1. Q.E.D.

330. COROLLARY. *The area of a trapezoid is equal to the product of the line joining the mid-points of its nonparallel sides by its altitude.*

How is the line joining the mid-points of the nonparallel sides related to the sum of the bases (§ 137)?

331. Area of an Irregular Polygon. The area of an irregular polygon may be found by dividing the polygon into triangles, and then finding the area of each of these triangles separately.



A common method used in land surveying is as follows: Draw the longest diagonal, and let fall perpendiculars upon this diagonal from the other vertices of the polygon. The sum of the right triangles, rectangles, and trapezoids is equivalent to the polygon.

EXERCISE 51

Find the areas of the parallelograms whose bases and altitudes are respectively as follows:

1. 2.25 in., $1\frac{1}{3}$ in.
2. 3.44 in., $1\frac{1}{2}$ in.
3. 2.7 ft., 1.2 ft.
4. 5.6 ft., 2.3 ft.
5. 2 ft. 3 in., 7 in.
6. 3 ft. 6 in., 2 ft.

Find the areas of the triangles whose bases and altitudes are respectively as follows:

7. 1.4 in., $1\frac{1}{2}$ in.
8. 2.5 in., 0.8 in.
9. $6\frac{1}{2}$ ft., 3 ft.
10. 5.4 ft., 1.2 ft.
11. 1 ft. 6 in., 8 in.
12. 3 ft. 8 in., 3 ft.

Find the areas of the trapezoids whose bases are the first two of the following numbers, and whose altitudes are the third numbers:

13. 2 ft., 1 ft., 6 in.
14. $2\frac{1}{2}$ ft., $1\frac{1}{4}$ ft., 9 in.
15. 3 ft. 7 in., 2 ft., 14 in.
16. 5 ft. 6 in., 3 ft., 2 ft.

Find the altitudes of the parallelograms whose areas and bases are respectively as follows:

17. 10 sq. in., 5 in.
18. 6 sq. in., 6 in.
19. 28 sq. ft., 7 ft.
20. 27 sq. ft., 6 ft.
21. 30 sq. ft., 12 ft.
22. 80 sq. in., 16 in.

Find the altitudes of the triangles whose areas and bases are respectively as follows:

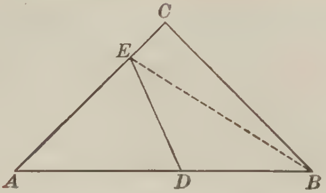
23. 49 sq. in., 14 in.
24. 48 sq. in., 12 in.
25. 50 sq. ft., 10 ft.
26. 160 sq. ft., 20 ft.
27. 110 sq. yd., 10 yd.
28. 176 sq. yd., 32 yd.

Find the altitudes of the trapezoids whose areas and bases are respectively as follows:

29. 33 sq. in., 5 in., 6 in.
30. 15 sq. in., 4 in., 6 in.
31. 13 sq. ft., 9 ft., 5 ft.
32. 70 sq. yd., 9 yd., 11 yd.

PROPOSITION VII. THEOREM

332. *The areas of two triangles that have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Given the triangles ABC and ADE, with the common angle A.

To prove that $\frac{\triangle ABC}{\triangle ADE} = \frac{AB \times AC}{AD \times AE}$.

Proof. Draw BE.

Then $\frac{\triangle ABC}{\triangle ABE} = \frac{AC}{AE}$,

and $\frac{\triangle ABE}{\triangle ADE} = \frac{AB}{AD}$. § 327

(Triangles having equal altitudes are to each other as their bases.)

Since we are considering numerical measures of area and length, we may treat all of the terms of these proportions as numbers.

Taking the product of the first members and the product of the second members of these equations, we have

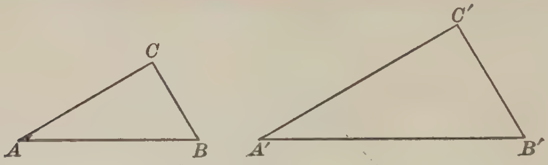
$$\frac{\triangle ABE \times \triangle ABC}{\triangle ADE \times \triangle ABE} = \frac{AB \times AC}{AD \times AE}. \quad \text{Ax. 3}$$

That is, by canceling $\triangle ABE$, we have the proportion

$$\frac{\triangle ABC}{\triangle ADE} = \frac{AB \times AC}{AD \times AE}. \quad \text{Q. E. D.}$$

PROPOSITION VIII. THEOREM

333. *The areas of two similar triangles are to each other as the squares on any two corresponding sides.*



Given the similar triangles ABC and $A'B'C'$.

To prove that

$$\frac{\triangle ABC}{\triangle A'B'C'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. Since the triangles are similar, Given

$$\therefore \angle A=\angle A'. \qquad \S\ 282$$

Then

$$\frac{\triangle ABC}{\triangle A'B'C'}=\frac{AB\times AC}{A'B'\times A'C'}. \qquad \S\ 332$$

(*The areas of two triangles that have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.*)

That is,

$$\frac{\triangle ABC}{\triangle A'B'C'}=\frac{AB}{A'B'}\times\frac{AC}{A'C'}.$$

But

$$\frac{AB}{A'B'}=\frac{AC}{A'C'}. \qquad \S\ 282$$

(*Similar polygons have their corresponding sides proportional.*)

Substituting $\frac{AB}{A'B'}$ for its equal $\frac{AC}{A'C'}$, we have

$$\frac{\triangle ABC}{\triangle A'B'C'}=\frac{AB}{A'B'}\times\frac{AB}{A'B'}, \qquad \text{Ax. 9}$$

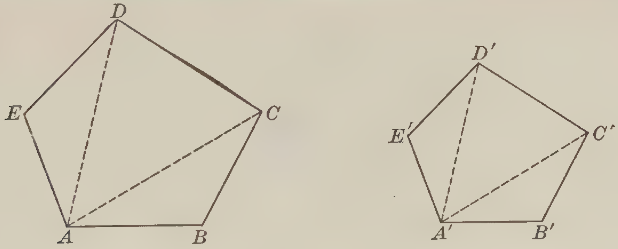
or

$$\frac{\triangle ABC}{\triangle A'B'C'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Q.E.D.

PROPOSITION IX. THEOREM

334. *The areas of two similar polygons are to each other as the squares on any two corresponding sides.*



Given the similar polygons *ABCDE* and *A′B′C′D′E′*, of area *s* and *s′* respectively.

To prove that $s : s' = \overline{AB}^2 : \overline{A'B'}^2$.

Proof. By drawing all the diagonals from any corresponding vertices A and A', the two similar polygons are divided into similar triangles. § 292

$$\therefore \frac{\triangle ADE}{\triangle A'D'E'} = \frac{\overline{AD}^2}{\overline{A'D'}^2} = \frac{\triangle ACD}{\triangle A'C'D'} = \frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad \text{§ 333}$$

That is,
$$\frac{\triangle ADE}{\triangle A'D'E'} = \frac{\triangle ACD}{\triangle A'C'D'} = \frac{\triangle ABC}{\triangle A'B'C'}. \quad \text{Ax. 8}$$

$$\therefore \frac{\triangle ADE + \triangle ACD + \triangle ABC}{\triangle A'D'E' + \triangle A'C'D' + \triangle A'B'C'} = \frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad \text{§ 269}$$

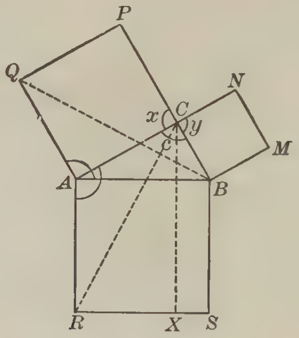
$\therefore s : s' = \overline{AB}^2 : \overline{A'B'}^2$, by Ax. 11. Q.E.D.

335. COROLLARY 1. *The areas of two similar polygons are to each other as the squares on any two corresponding lines.*

336. COROLLARY 2. *Corresponding sides of two similar polygons have the same ratio as the square roots of the areas.*

Proposition X. Theorem

337. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.*



Given the right triangle ABC, with AS the square on the hypotenuse, and BN, CQ the squares on the other two sides.

To prove that $AS = BN + CQ$.

Proof. Draw CX through $C \parallel$ to BS. § 233

Draw CR and BQ.

Since $\angle$s c and x are rt. $\angle$s, the $\angle PCB$ is a straight angle, § 34

and the line PCB is a straight line. § 43

Similarly, the line ACN is a straight line.

In the $\triangle$s ARC and ABQ,

$$AR = AB,$$

$$AC = AQ, \quad \text{§ 65}$$

and $$\angle RAC = \angle BAQ. \quad \text{Ax. 1}$$

(*Each is the sum of a rt. $\angle$ and the $\angle BAC$.*)

$\therefore \triangle ARC$ is congruent to $\triangle ABQ$. § 68

Furthermore the $\square AX$ is double the $\triangle ARC$. § 325

(*They have the same base AR, and the same altitude RX.*)

Again the square CQ is double the $\triangle ABQ$. § 325

(*They have the same base* AQ, *and the same altitude* AC.)

$\therefore$ the $\square AX$ is equivalent to the square CQ. Ax. 3

In like manner, by drawing CS and AM, it may be proved that the rectangle BX is equivalent to the square BN.

Since square $AS = \square BX + \square AX$, Ax. 11

$\therefore AS = BN + CQ$, by Ax. 9. Q.E.D

The first proof of this theorem is usually attributed to Pythagoras (about 525 B.C.), although the truth of the proposition was known earlier. It is one of the most important propositions of geometry. Various proofs may be given, but the one here used is the most common. This proof is attributed to Euclid (about 300 B.C.), a famous Greek geometer.

338. Corollary 1. *The square on either side of a right triangle is equivalent to the difference of the square on the hypotenuse and the square on the other side.*

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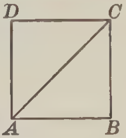
339. Corollary 2. *The diagonal and a side of a square are incommensurable.*



For $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AB}^2$.

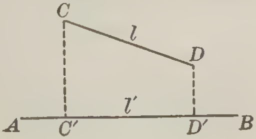
$\therefore AC = AB\sqrt{2}$.

Since $\sqrt{2}$ may be carried to as many decimal places as we please, but cannot be exactly expressed as a rational fraction, it has no common measure with 1. That is, $AC : AB = \sqrt{2}$, an incommensurable number.

340. Projection. If from the extremities of a line-segment perpendiculars are let fall upon another line, the segment thus cut off is called the *projection* of the first line upon the second.

Thus $C'D'$ is the projection of CD upon AB, or l' is the projection of l upon AB.

In general it is convenient to designate by the small letter a the side of a triangle opposite $\angle A$, and so for the other sides; to designate the projection of a by a'; and to designate the height (altitude) by h.



EXERCISE 52

Given the sides of a right triangle as follows, find the hypotenuse to two decimal places:

1. 30 ft., 40 ft.

2. 45 ft., 60 ft.

3. 20 ft., 30 ft.

4. 1.5 in., 2.5 in.

5. 2 ft. 6 in., 3 ft.

6. 3 ft. 8 in., 2 ft.

Given the hypotenuse and one side of a right triangle as follows, find the other side to two decimal places:

7. 50 ft., 40 ft.

8. 35 ft., 21 ft.

9. 10 ft., 6 ft.

10. 1.2 in., 0.8 in.

11. 3 ft. 4 in., 2 ft.

12. 6 ft. 2 in., 5 ft.

13. A ladder 38 ft. 6 in. long is placed against a wall, with its foot 23.1 ft. from the base of the wall. How high does it reach on the wall?

14. Find the altitude of an equilateral triangle with side s.

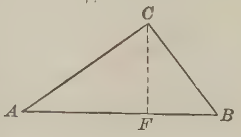
15. Find the side of an equilateral triangle with altitude h.

16. The area of an equilateral triangle with side s is $\frac{1}{4} s^2 \sqrt{3}$.

17. Find the length of the longest chord and of the shortest chord that can be drawn through a point 1 ft. from the center of a circle whose radius is 20 in.

18. The radius of a circle is 5 in. Through a point 3 in. from the center a diameter is drawn, and also a chord perpendicular to the diameter. Find the length of this chord, and the distance (to two decimal places) from one end of the chord to the ends of the diameter.

19. In this figure the angle C is a right angle. From the relations $\overline{AC}^2 = AB \times AF$ (§ 294) and $\overline{CB}^2 = AB \times BF$, show that $\overline{AC}^2 + \overline{CB}^2 = \overline{AB}^2$.



20. If the diagonals of a quadrilateral intersect at right angles, the sum of the squares on one pair of opposite sides is equivalent to the sum of the squares on the other pair.

[illegible] [illegible]

PROPOSITION XI. THEOREM

[illegible] [illegible]

341. *In any triangle the square on the side opposite an acute angle is equivalent to the sum of the squares on the other two sides diminished by twice the product of one of those sides by the projection of the other upon that side.*

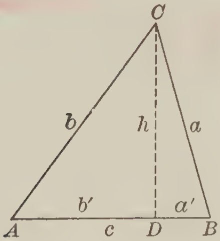


FIG. 1

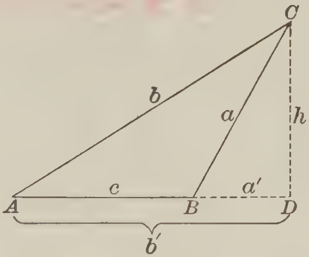


FIG. 2

Given the triangle ABC, A being an acute angle, and a' and b' being the projections of a and b respectively upon c.

To prove that $a^2 = b^2 + c^2 - 2\,b'c.$

Proof. If D, the foot of the $\perp$ from C, falls upon c (Fig. 1),

$$a' = c - b'.$$

If D falls upon c produced (Fig. 2),

$$a' = b' - c.$$

In either case, by squaring, we have

$$a'^2 = b'^2 + c^2 - 2\,b'c. \qquad \text{Ax. 5}$$

Adding h^2 to each side of this equation, we have

$$h^2 + a'^2 = h^2 + b'^2 + c^2 - 2\,b'c. \qquad \text{Ax. 1}$$

But $h^2 + a'^2 = a^2$, and $h^2 + b'^2 = b^2$. § 337

Putting a^2 and b^2 for their equals in the above equation, we have $a^2 = b^2 + c^2 - 2\,b'c$, by Ax. 9. Q.E.D

Proposition XII. Theorem

342. *In any obtuse triangle the square on the side opposite the obtuse angle is equivalent to the sum of the squares on the other two sides increased by twice the product of one of those sides by the projection of the other upon that side.*

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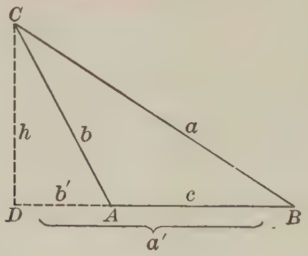


Given the obtuse triangle ABC, A being the obtuse angle, and a' and b' the projections of a and b respectively upon c.

To prove that $a^2 = b^2 + c^2 + 2\,b'c$.

Proof. $a' = b' + c$. Ax. 11

Squaring, $a'^2 = b'^2 + c^2 + 2\,b'c$. Ax. 5

Adding h^2 to each side of this equation, we have

$$h^2 + a'^2 = h^2 + b'^2 + c^2 + 2\,b'c. \quad \text{Ax. 1}$$

But $h^2 + a'^2 = a^2$, and $h^2 + b'^2 = b^2$. § 337

Putting a^2 and b^2 for their equals in the above equation, we have

$$a^2 = b^2 + c^2 + 2\,b'c, \text{ by Ax. 9.} \quad \text{Q. E. D.}$$

Discussion. By the Principle of Continuity the last three theorems may be included in one theorem by letting the $\angle A$ change from an acute angle to a right angle and then to an obtuse angle. Let the student explain.

The last three theorems enable us to compute the altitudes of a triangle if the three sides are known; for in Prop. XII we can find b', and from b and b' we can find h.

EXERCISE 53

Find the lengths, to two decimal places, of the diagonals of the squares whose sides are:

1. 7 in. **2.** 10 in. **3.** 9.2 in. **4.** 1 ft. 6 in. **5.** 2 ft. 3 in.

Find the lengths, to two decimal places, of the sides of the squares whose diagonals are:

6. 4 in. **7.** 8 in. **8.** 5 ft. **9.** $\sqrt{5}$ in. **10.** 2 ft. 6 in.

11. The minute hand and hour hand of a clock are 6 in. and $4\frac{1}{2}$ in. long respectively. How far apart are the ends of the hands at 9 o'clock?

12. A rectangle whose base is 9 and diagonal 15 has the same area as a square whose side is x. Find the value of x.

13. A ring is screwed into a ceiling in a room 10 ft. high. Two rings are screwed into the floor at points 5 ft. and 12 ft. from a point directly beneath the one in the ceiling. Wires are stretched from the ceiling ring to each floor ring. How long are the wires? (Answer to two decimal places.)

14. The sum of the squares on the segments of two perpendicular chords is equivalent to the square on the diameter of the circle.

If AB, CD are the chords, draw the diameter BE, and draw AC, ED, BD. Prove that $AC = ED$.

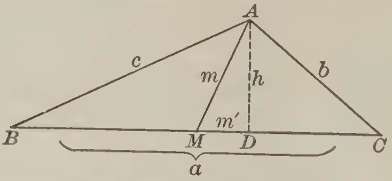
15. The difference of the squares on two sides of a triangle is equivalent to the difference of the squares on the segments of the third side, made by the perpendicular on the third side from the opposite vertex.

16. In an isosceles triangle the square on one of the equal sides is equivalent to the square on any line drawn from the vertex to the base, increased by the product of the segments of the base.

Proposition XIII. Theorem

343. *The sum of the squares on two sides of a triangle is equivalent to twice the square on half the third side, increased by twice the square on the median upon that side.*

The difference of the squares on two sides of a triangle is equivalent to twice the product of the third side by the projection of the median upon that side.



Given the triangle ABC, the median m, and m' the projection of m upon the side a. Also let c be greater than b.

To prove that 1. $c^2 + b^2 = 2\,\overline{BM}^2 + 2\,m^2$;

2. $c^2 - b^2 = 2\,am'$.

Proof. The $\angle AMB$ is obtuse, and the $\angle CMA$ is acute. § 116

Since $c > b$, M lies between B and D. § 84

Then $c^2 = \overline{BM}^2 + m^2 + 2\,BM \cdot m'$, § 342

and $b^2 = \overline{MC}^2 + m^2 - 2\,MC \cdot m'$. § 341

Adding these equals, and observing that $BM = MC$, we have

$c^2 + b^2 = 2\,\overline{BM}^2 + 2\,m^2$. Ax. 1

Subtracting the second from the first, we have

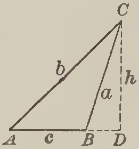
$c^2 - b^2 = 2\,am'$, by Ax. 2. Q.E.D.

Discussion. Consider the proposition when $c = b$.

This theorem may be omitted without interfering with the regular sequence. It enables us to compute the medians when the three sides are known.

EXERCISE 54

1. To compute the area of a triangle in terms of its sides.



At least one of the angles A or B is acute. Suppose A is acute.

In the $\triangle ADC$, $h^2 = b^2 - \overline{AD}^2$. Why?

In the $\triangle ABC$, $a^2 = b^2 + c^2 - 2c \times AD$. Why?

Therefore $AD = \dfrac{b^2 + c^2 - a^2}{2c}$.

Hence
$$h^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2} = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2}$$
$$= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4c^2}$$
$$= \frac{\{(b + c)^2 - a^2\}\{a^2 - (b - c)^2\}}{4c^2}$$
$$= \frac{(a + b + c)(b + c - a)(a + b - c)(a - b + c)}{4c^2}.$$

Let $a + b + c = 2s$, where s stands for semiperimeter.

Then $b + c - a = a + b + c - 2a = 2s - 2a = 2(s - a)$.

Similarly $a + b - c = 2(s - c)$,

and $a - b + c = 2(s - b)$.

Hence
$$h^2 = \frac{2s \times 2(s - a) \times 2(s - b) \times 2(s - c)}{4c^2}.$$

By simplifying, and extracting the square root,
$$h = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

Hence the area $= \frac{1}{2}ch = \sqrt{s(s - a)(s - b)(s - c)}$.

For example, if the sides are 3, 4, and 5,
$$\text{area} = \sqrt{6(6 - 3)(6 - 4)(6 - 5)} = \sqrt{6 \cdot 3 \cdot 2} = 6.$$

If Ex. 1 has been studied, find the areas, to two decimal places, of the triangles whose sides are:

2. 4, 5, 6.

3. 5, 6, 7.

4. 6, 8, 10.

5. 6, 8, 9.

6. 7, 8, 11.

7. 9, 10, 11.

8. 1.2, 3, 2.1.

9. 11, 12, 13.

10. To compute the radius of the circle circumscribed about a triangle in terms of the sides of the triangle. (Solve only if § 305 and Ex. 1 have been taken.)

Let CD be a diameter.

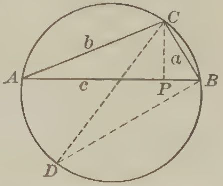
By § 305, what do we know about the products $CA \times BC$ and $CD \times CP$?

What does this tell us of ab and $2r \cdot CP$, r being the radius?

From Ex. 1, what does CP equal in terms of the sides?

Is it therefore possible to show that

$$r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}?$$



If Exs. 1 and 10 have been studied, compute the radii, to two decimal places, of the circles circumscribed about the triangles whose sides are:

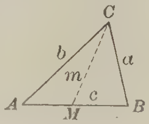
11. 3, 4, 5. **12.** 27, 36, 45. **13.** 7, 9, 11. **14.** 10, 11, 12

15. To compute the medians of a triangle in terms of its sides.

Omit if § 343 has not been taken. What do we know about $a^2 + b^2$ as compared with $2m^2 + 2\left(\frac{c}{2}\right)^2$?

From this relation show that

$$m = \tfrac{1}{2}\sqrt{2(a^2 + b^2) - c^2}.$$



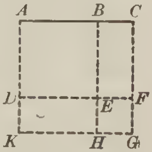
If Ex. 15 has been studied, compute the three medians, to two decimal places, of the triangles whose sides are:

16. 3, 4, 5. **17.** 6, 8, 10. **18.** 6, 7, 8. **19.** 7, 9, 11.

20. If the sides of a triangle are 7, 9, and 11, is the angle opposite the side 11 right, acute, or obtuse?

21. The square constructed upon the sum of two lines is equivalent to the sum of the squares constructed upon these two lines, increased by twice the rectangle of these lines.

Given the two lines *AB* and *BC*, and *AC* their sum. Construct the squares *AKGC* and *ADEB* upon *AC* and *AB* respectively. Produce *BE* and *DE* to meet *KG* and *CG* in *H* and *F* respectively. Then we have the square *EHGF*, with sides each equal to *BC*. Hence the square *AKGC* is the sum of the squares *ADEB* and *EHGF*, and the rectangles *DKHE* and *BEFC*.

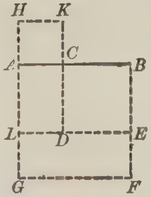


This proves geometrically the algebraic formula

$$(a+b)^2 = a^2 + 2\,ab + b^2.$$

22. The square constructed upon the difference of two lines is equivalent to the sum of the squares constructed upon these two lines, diminished by twice their rectangle.

Given the two lines *AB* and *AC*, and *BC* their difference. Construct the square *AGFB* upon *AB*, the square *ACKH* upon *AC*, and the square *CDEB* upon *BC*. Produce *ED* to meet *AG* in *L*. The dimensions of the rectangles *LGFE* and *HLDK* are *AB* and *AC*, and the square *CDEB* is the difference between the whole figure and the sum of these rectangles.

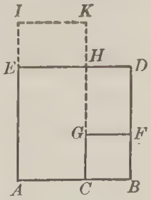


This proves geometrically the algebraic formula

$$(a-b)^2 = a^2 - 2\,ab + b^2.$$

23. The difference between the squares constructed upon two lines is equivalent to the rectangle of the sum and difference of these lines.

Given the squares *ABDE* and *CBFG*, constructed upon *AB* and *BC*. The difference between these squares is the polygon *ACGFDE*, which is composed of the rectangles *ACHE* and *GFDH*. Produce *AE* and *CH* to *I* and *K* respectively, making *EI* and *HK* each equal to *BC*, and draw *IK*. The difference between the squares *ABDE* and *CBFG* is then equivalent to the rectangle *ACKI*, with dimensions *AB* + *BC*, and *AB* – *BC*

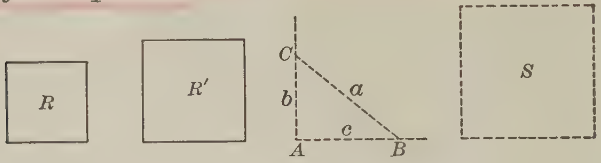


This proves geometrically the algebraic formula

$$a^2 - b^2 = (a+b)(a-b).$$

Proposition XIV. Problem

344. *To construct a square equivalent to the sum of two given squares.*



Given the two squares, R and R'.

Required to construct a square equivalent to $R + R'$.

Construction. Construct the rt. $\angle A$. § 228

On the sides of $\angle A$, take AB, or c, equal to a side of R', and AC, or b, equal to a side of R, and draw BC, or a.

Construct the square S, having a side equal to BC.

Then S is the square required. Q.E.F.

Proof. $a^2 = b^2 + c^2$. § 337

(*The square on the hypotenuse of a rt.* △ *is equivalent to the sum of the squares on the other two sides.*)

$\therefore\ S = R + R'$, by Ax. 9. Q.E.D.

345. Corollary 1. *To construct a square equivalent to the difference of two given squares.*

We may easily reverse the above construction by first drawing c, then erecting a ⊥ at A, and then with a radius a fixing the point C.

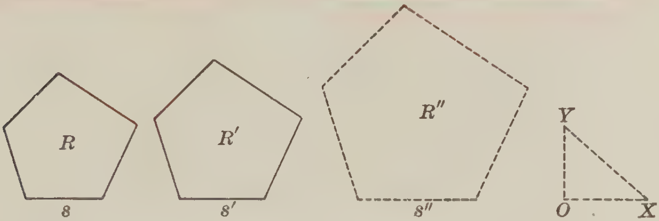
346. Corollary 2. *To construct a square equivalent to the sum of three given squares.*

If a side of the third square is d, we may erect a perpendicular from C to the line BC, take CD equal to d, and join D and B.

Discussion. It is evident that we can continue this process indefinitely, and thus construct a square equivalent to the sum of any number of given squares.

Proposition XV. Problem

347. *To construct a polygon similar to two given similar polygons and equivalent to their sum.*



Given the two similar polygons R and R'.

Required to construct a polygon similar to R and R', and equivalent to $R + R'$.

Construction. Construct the rt. ∠ O. § 228

Let s and s' be corresponding sides of R and R'.

On the sides of ∠O, take OX equal to s', and OY equal to s.

Draw XY, and take s'' equal to XY.

Upon s'', corresponding to s, construct R'' similar to R. § 312

Then R'' is the polygon required. Q.E.F.

Proof.
$$\overline{OY}^2 + \overline{OX}^2 = \overline{XY}^2. \qquad \text{§ 337}$$

Putting for OY, OX, and XY their equals s, s', and s'', we have

$$s^2 + s'^2 = s''^2. \qquad \text{Ax. 9}$$

But
$$\frac{R}{R''} = \frac{s^2}{s''^2},$$

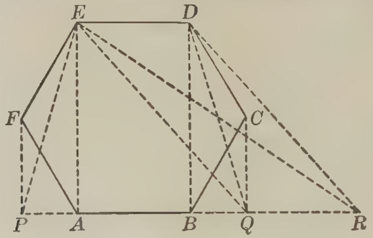
and
$$\frac{R'}{R''} = \frac{s'^2}{s''^2}. \qquad \text{§ 334}$$

By addition,
$$\frac{R + R'}{R''} = \frac{s^2 + s'^2}{s''^2} = 1. \qquad \text{Ax. 1}$$

$$\therefore R'' = R + R', \text{ by Ax. 3.} \qquad \text{Q.E.D.}$$

PROPOSITION XVI. PROBLEM

348. *To construct a triangle equivalent to a given polygon.*



Given the polygon *ABCDEF*.

Required to construct a triangle equivalent to ABCDEF

Construction. Let B, C, and D be any three consecutive vertices of the polygon. Draw the diagonal DB.

From C draw a line ∥ to DB. § 233

Produce AB to meet this line at Q, and draw DQ.

Again, draw EQ, and from D draw a line ∥ to EQ, meeting AB produced at R, and draw ER.

In like manner continue to reduce the number of sides of the polygon until we obtain the $\triangle EPR$.

Then $\triangle EPR$ is the triangle required. Q. E. F.

Proof. The polygon $AQDEF$ has one side less than the polygon $ABCDEF$.

Furthermore, in the two polygons, the part $ABDEF$ is common,

and the $\triangle BQD = \triangle BCD$. § 326

(*For the base DB is common, and their vertices C and Q are in the line CQ ∥ to the base.*)

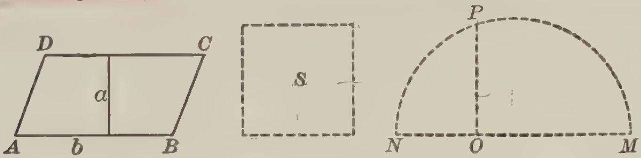
$\therefore AQDEF = ABCDEF$. Ax. 1

In like manner it may be proved that

$AREF = AQDEF$, and $EPR = AREF$. Q. E. D.

PROPOSITION XVII. PROBLEM

349. *To construct a square equivalent to a given parallelogram.*



Given the parallelogram *ABCD.*

Required to construct a square equivalent to the $\square ABCD$.

Construction. Upon any convenient line take NO equal to a, and OM equal to b, the altitude and base respectively of $\square ABCD$.

Upon NM as a diameter describe a semicircle.

At O erect $OP \perp$ to NM, meeting the circle at P. § 228

Construct the square S, having a side equal to OP.

Then S is the square required. Q. E. F.

Proof. $NO : OP = OP : OM.$ § 297

$\therefore \overline{OP}^2 = NO \times OM.$ § 261

That is, $\overline{OP}^2 = ab.$ Ax. 9

But $S = \overline{OP}^2,$

and $\square ABCD = ab.$ § 322

$\therefore S = \square ABCD$, by Ax. 9. Q. E. D.

350. COROLLARY 1. *To construct a square equivalent to a given triangle.*

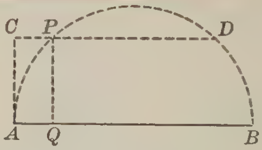
Take for a side of the square the mean proportional between the base and half the altitude of the triangle.

351. COROLLARY 2. *To construct a square equivalent to a given polygon.*

First reduce the polygon to an equivalent triangle, and then construct a square equivalent to the triangle.

PROPOSITION XVIII. PROBLEM

352. *To construct a parallelogram equivalent to a given square, and having the sum of its base and altitude equal to a given line.*



Given the square *S*, and the line *AB*.

Required to construct a ▱ equivalent to S, with the sum of its base and altitude equal to AB.

Construction. Upon AB as a diameter describe a semicircle.

At A erect $AC \perp$ to AB and equal to a side of the given square S. § 228

Draw $CD \parallel$ to AB, cutting the circle at P. § 233

Draw $PQ \perp$ to AB. § 227

Then any ▱, as P, having AQ for its altitude and QB for its base is equivalent to S. Q.E.F.

Proof. $AQ : PQ = PQ : QB$. § 297

$\therefore \overline{PQ}^2 = AQ \times QB$. § 261

Furthermore PQ is $\parallel$ to CA. § 95

$\therefore PQ = CA$. § 127

$\therefore \overline{PQ}^2 = \overline{CA}^2$. Ax. 5

$\therefore AQ \times QB = \overline{CA}^2$. Ax. 8

But $P = AQ \times QB$, § 322

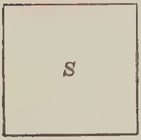
and $S = \overline{CA}^2$. § 320

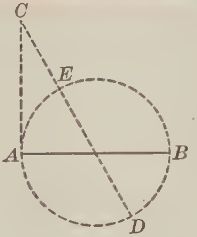
$\therefore P = S$, by Ax. 8. Q.E.D.

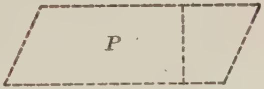
Thus is solved geometrically the algebraic problem, given $x + y = a$, $xy = b$, to find x and y.

PROPOSITION XIX. PROBLEM

353. *To construct a parallelogram equivalent to a given square, and having the difference of its base and altitude equal to a given line.*







Given the square S, and the line AB.

Required to construct a ▱ equivalent to S, with the difference of its base and altitude equal to AB.

Construction. Upon AB as a diameter describe a circle.

From A draw AC, tangent to the circle, § 246
and equal to a side of the given square S.

Through the center of the circle draw CD intersecting the circle at E and D.

Then any ▱, as P, having CD for its base and CE for its altitude, is equivalent to S. **Q. E. F.**

Proof. $CD : CA = CA : CE.$ § 302

$\therefore \overline{CA}^2 = CD \times CE,$ § 261

and the difference between CD and CE is the diameter of the circle, that is, AB.

But $P = CD \times CE,$ § 322

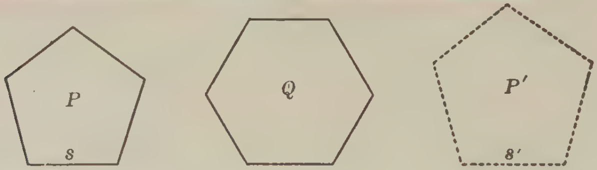
and $S = \overline{CA}^2.$ § 320

$\therefore P = S$, by Ax. 8. **Q. E. D.**

Thus is solved geometrically the algebraic problem, given $x - y = a$, $xy = b$, to find x and y.

Proposition XX. Problem

354. *To construct a polygon similar to a given polygon and equivalent to another given polygon.*



Given the polygons P and Q.

Required to construct a polygon similar to P and equivalent to Q.

Construction. Construct squares equivalent to P and Q, § 351
and let m and n respectively denote their sides.

Let s be any side of P.

Find s', the fourth proportional to m, n, and s. § 307

Upon s', corresponding to s,
construct a polygon P' similar to the polygon P. § 312

Then P' is the polygon required. Q.E.F.

Proof. Since $m : n = s : s'$, Const.

$\therefore m^2 : n^2 = s^2 : s'^2$. § 270

But $P = m^2$, and $Q = n^2$. Const.

$\therefore P : Q = s^2 : s'^2$. Ax. 9

But $P : P' = s^2 : s'^2$. § 334

(*The areas of two similar polygons are to each other as the squares on any two corresponding sides.*)

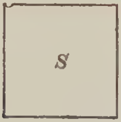
$\therefore P : Q = P : P'$. Ax. 8

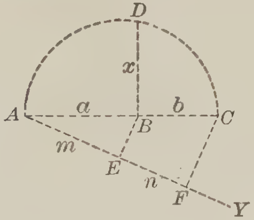
$\therefore P' = Q$. § 263

$\therefore P'$, being similar to P, is the polygon required. Q.E.D.

PROPOSITION XXI. PROBLEM

355. *To construct a square which shall have a given ratio to a given square.*





Given the square S, and the ratio $\frac{n}{m}$.

Required to construct a square which shall be to S as n is to m.

Construction. Take AB equal to a side of S, and draw AY, making any convenient angle with AB.

On AY take AE equal to m units and EF equal to n units.

Draw EB.

From F draw a line ∥ to EB, meeting AB produced at C. § 233

On AC as a diameter describe a semicircle.

At B erect $BD \perp$ to AC, meeting the semicircle at D. § 228

Then BD is a side of the square required. Q. E. F.

Proof. Denote AB by a, BC by b, and BD by x.

Then $a : x = x : b$. § 297

$\therefore a : b = a^2 : x^2$. § 271

But $a : b = m : n$. § 273

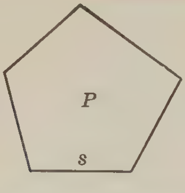
$\therefore a^2 : x^2 = m : n$. Ax. 8

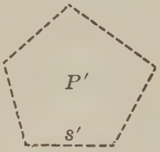
By inversion, $x^2 : a^2 = n : m$. § 266

Hence the square on BD will have the same ratio to S as n has to m. Q. E. D.

PROPOSITION XXII. PROBLEM

356. *To construct a polygon similar to a given polygon and having a given ratio to it.*





Given the polygon P and the ratio $\frac{n}{m}$.

Required to construct a polygon similar to P, which shall be to P as n is to m.

Construction. Let s be any side of P.

Draw a line s', such that the square on s' shall be to the square on s as n is to m. § 355

Upon s' as a side corresponding to s construct the polygon P' similar to P. § 312

(*Upon a given line corresponding to a given side of a given polygon, to construct a polygon similar to the given polygon.*)

Then P' is the polygon required. Q.E.F.

Proof. $P' : P = s'^2 : s^2$. § 334

(*The areas of two similar polygons are to each other as the squares on any two corresponding sides.*)

But $s'^2 : s^2 = n : m$. Const.

Therefore $P' : P = n : m$, by Ax. 8. Q.E.D.

This problem enables us to construct a square that is twice a given square or half a given square, to construct an equilateral triangle that shall be any number of times a given equilateral triangle, and in general to enlarge or to reduce any figure in a given ratio. An architect's drawing, for example, might need to be enlarged so as to be double the area of the original, and the scale could be found by this method.

EXERCISE 55

PROBLEMS OF COMPUTATION

1. The sides of a triangle are 0.7 in., 0.6 in., and 0.7 in. respectively. Is the largest angle acute, right, or obtuse?

2. The sides of a triangle are 5.1 in., 6.8 in., and 8.5 in. respectively. Is the largest angle acute, right, or obtuse?

3. Find the area of an isosceles triangle whose perimeter is 14 in. and base 4 in. (One decimal place.)

4. Find the area of an equilateral triangle whose perimeter is 18 in. (One decimal place.)

5. Find the area of a right triangle, the hypotenuse being 1.7 in. and one of the other sides being 0.8 in.

6. Find the ratio of the altitudes of two triangles of equal area, the base of one being 1.5 in. and that of the other 4.5 in.

7. The bases of a trapezoid are 34 in. and 30 in., and the altitude is 2 in. Find the side of a square having the same area.

8. What is the area of the isosceles right triangle in which the hypotenuse is $\sqrt{2}$?

9. What is the area of the isosceles right triangle in which the hypotenuse is $7\sqrt{2}$?

10. If the side of an equilateral triangle is $2\sqrt{3}$, what is the altitude of the triangle? the area of the triangle?

11. If the side of an equilateral triangle is 1 ft., what is the area of the triangle?

12. If the area of an equilateral triangle is 43.3 sq. in., what is the base of the triangle? (Take $\sqrt{3} = 1.732$.)

13. The sides of a triangle are 2.8 in., 3.5 in., and 2.1 in. respectively. Draw the figure carefully and see what kind of a triangle it is. Verify this conclusion by applying a geometric test, and find the area of the triangle.

EXERCISE 56

THEOREMS

1. The area of a rhombus is equal to half the product of its diagonals.

2. Two triangles are equivalent if the base of the first is equal to half the altitude of the second, and the altitude of the first is equal to twice the base of the second.

3. The area of a circumscribed polygon is equal to half the product of its perimeter by the radius of the inscribed circle.

4. Two parallelograms are equivalent if their altitudes are reciprocally proportional to their bases.

5. If equilateral triangles are constructed on the sides of a right triangle, the triangle on the hypotenuse is equivalent to the sum of the triangles on the other two sides.

6. If similar polygons are constructed on the sides of a right triangle, as corresponding sides, the polygon on the hypotenuse is equivalent to the sum of the polygons on the other two sides.

Ex. 6 is one of the general forms of the Pythagorean Theorem.

7. If lines are drawn from any point within a parallelogram to the four vertices, the sum of either pair of triangles with parallel bases is equivalent to the sum of the other pair.

8. Every line drawn through the intersection of the diagonals of a parallelogram bisects the parallelogram.

9. The line that bisects the bases of a trapezoid divides the trapezoid into two equivalent parts.

10. If a quadrilateral with two sides parallel is bisected by either diagonal, the quadrilateral is a parallelogram.

11. The triangle formed by two lines drawn from the midpoint of either of the nonparallel sides of a trapezoid to the opposite vertices is equivalent to half the trapezoid.

EXERCISE 57

PROBLEMS OF CONSTRUCTION

1. Given a square, to construct a square of half its area.

2. To construct a right triangle equivalent to a given oblique triangle.

3. To construct a triangle equivalent to the sum of two given triangles.

4. To construct a triangle equivalent to a given triangle, and having one side equal to a given line.

5. To construct a rectangle equivalent to a given parallelogram, and having its altitude equal to a given line.

6. To construct a right triangle equivalent to a given triangle, and having one of the sides of the right angle equal to a given line.

7. To construct a right triangle equivalent to a given triangle, and having its hypotenuse equal to a given line.

8. To divide a given triangle into two equivalent parts by a line through a given point P in the base.

9. To draw from a given point P in the base AB of a triangle ABC a line to AC produced, so that it may be bisected by BC.

10. To find a point within a given triangle such that the lines from this point to the vertices shall divide the triangle into three equivalent triangles.

11. To divide a given triangle into two equivalent parts by a line parallel to one of the sides.

12. Through a given point to draw a line so that the segments intercepted between the point and perpendiculars drawn to the line from two other given points may have a given ratio.

13. To find a point such that the perpendiculars from it to the sides of a given triangle shall be in the ratio p, q, r

EXERCISE 58

REVIEW QUESTIONS

1. What is meant by the area of a surface? Illustrate.

2. What is the difference between equivalent figures and congruent figures?

3. State two propositions relating to the ratio of one rectangle to another.

4. Given the base and altitude of a rectangle, how is the area found? Given the area and base, how is the altitude found?

5. How do you justify the expression, "the product of two lines"? "the quotient of an area by a line"?

6. Can a triangle with a perimeter of 10 in. have the same area as one with a perimeter of 1 in.? Is the same answer true for two squares?

7. Can a parallelogram with a perimeter of 10 in. have the same area as a rectangle with a perimeter of 1 in.? Is the same answer true for two rectangles?

8. Explain how the area of an irregular field with straight sides may be found by the use of the theorems of Book IV.

9. A triangle has two sides 5 and 6, including an angle of 70°, and another triangle has two sides 2 and $7\frac{1}{2}$, including an angle of 70°. What is the ratio of the areas of the triangles?

10. Two similar triangles have two corresponding sides 5 in. and 15 in. respectively. The larger triangle has how many times the area of the smaller?

11. Given the hypotenuse of an isosceles right triangle, how do you proceed to find the area?

12. Given three sides of a triangle, what test can you apply to determine whether or not it is a right triangle?

13. Suppose you wish to construct a square equivalent to a given polygon, how do you proceed?

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BOOK V

REGULAR POLYGONS AND CIRCLES

357. Regular Polygon. A polygon that is both equiangular and equilateral is called a *regular polygon.*

Familiar examples of regular polygons are the equilateral triangle and the square.

It is proved in Prop. I (§ 362) that a circle may be circumscribed about, and a circle may be inscribed in, any regular polygon, and that these circles are concentric (§ 188).

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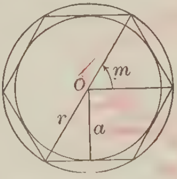


358. Radius. The radius of the circle circumscribed about a regular polygon is called the *radius* of the polygon.

In this figure r is the radius of the polygon.

359. Apothem. The radius of the circle inscribed in a regular polygon is called the *apothem* of the polygon.

In the figure a is the apothem of the polygon. The apothem is evidently perpendicular to the side of the regular polygon (§ 185).

360. Center. The common center of the circles circumscribed about and inscribed in a regular polygon is called the *center* of the polygon.

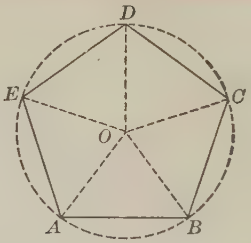
In the figure O is the center of the polygon.

361. Angle at the Center. The angle between the radii drawn to the extremities of any side of a regular polygon is called the *angle at the center* of the polygon.

In the figure m is the angle at the center of the polygon. It is evidently subtended by the chord which is the side of the inscribed polygon.

PROPOSITION I. THEOREM

362. *A circle may be circumscribed about, and a circle may be inscribed in, any regular polygon.*



Given the regular polygon *ABCDE*.

To prove that 1. *a circle may be circumscribed about $ABCDE$;*
2. *a circle may be inscribed in $ABCDE$.*

Proof. 1. Let O be the center of the circle which may be passed through the three vertices A, B, and C. § 190

Draw OA, OB, OC, OD.

Then $OB = OC$, § 162

and $AB = CD$. § 357

Furthermore $\angle CBA = \angle DCB$, § 357

and $\angle CBO = \angle OCB$. § 74

$\therefore \angle OBA = \angle DCO$. Ax. 2

$\therefore \triangle OAB$ is congruent to $\triangle OCD$. § 68

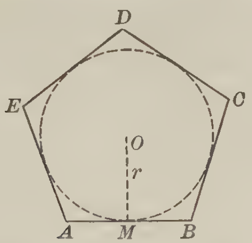
$\therefore OA = OD$. § 67

Therefore the circle that passes through A, B, C, passes also through D.

In like manner it may be proved that the circle that passes through B, C, and D passes also through E; and so on.

Therefore the circle described with O as a center and OA as a radius will be circumscribed about the polygon, by § 205. Q.E.D.

Proof. 2. Let O be the center of the circumscribed circle.



Since the sides of the regular polygon are equal chords of the circumscribed circle, they are equally distant from the center. § 178

Therefore the circle described with O as a center, with the perpendicular from O to a side of the polygon as a radius, will be inscribed in the polygon, by § 205. Q. E. D.

363. Corollary 1. *The radius drawn to any vertex of a regular polygon bisects the angle at the vertex.*

364. Corollary 2. *The angles at the center of any regular polygon are equal, and each is supplementary to an interior angle of the polygon.*

For the angles at the center are corresponding angles of congruent triangles. If M is the mid-point of AB, then since the $\angle$s MOB and OBM are complementary what can we say of their doubles, AOB and CBA?

365. Corollary 3. *An equilateral polygon inscribed in a circle is a regular polygon.*

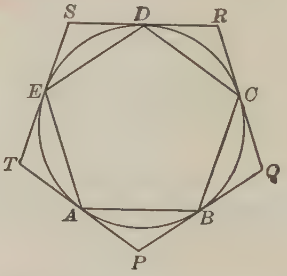
Why are the angles also equal?

366. Corollary 4. *An equiangular polygon circumscribed about a circle is a regular polygon.*

By joining consecutive points of contact of the sides of the polygon can you show that certain isosceles triangles are congruent, and thus prove the polygon equilateral?

Proposition II. Theorem

367. *If a circle is divided into any number of equal arcs, the chords joining the successive points of division form a regular inscribed polygon; and the tangents drawn at the points of division form a regular circumscribed polygon.*



Given a circle divided into equal arcs by *A*, *B*, *C*, *D*, and *E*, *AB*, *BC*, *CD*, *DE*, and *EA* being chords, and *PQ*, *QR*, *RS*, *ST*, and *TP* being tangents at *B*, *C*, *D*, *E*, and *A* respectively.

To prove that 1. *ABCDE is a regular polygon;*

2. *PQRST is a regular polygon.*

Proof. 1. Since the arcs are equal by construction,

$\therefore AB = BC = CD = DE = EA.$ § 170

$\therefore ABCDE$ is a regular polygon. § 365

(*An equilateral polygon inscribed in a circle is a regular polygon.*)

Proof. 2. $\angle P = \angle Q = \angle R = \angle S = \angle T.$ § 221

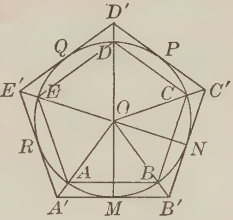
(*An $\angle$ formed by two tangents is measured by half the difference of the intercepted arcs.*)

$\therefore PQRST$ is a regular polygon. § 366

(*An equiangular polygon circumscribed about a circle is a regular polygon.*) **Q.E.D.**

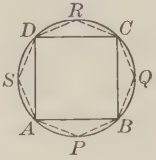
368. Corollary 1. *Tangents to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.*

369. Corollary 2. *Tangents to a circle at the mid-points of the arcs subtended by the sides of a regular inscribed polygon form a regular circumscribed polygon, whose sides are parallel to the sides of the inscribed polygon and whose vertices lie on the radii (produced) of the inscribed polygon.*

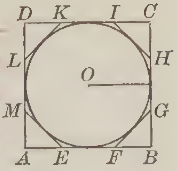


For two corresponding sides, AB and $A'B'$, are perpendicular to OM (§§ 176, 185), and are parallel (§ 95); and the tangents MB' and NB', intersecting at a point equidistant from OM and ON (§ 192), intersect upon the bisector of the $\angle MON$ (§ 152); that is, upon the radius OB (§ 363).

370. Corollary 3. *Lines drawn from each vertex of a regular polygon to the mid-points of the adjacent arcs subtended by the sides of the polygon form a regular inscribed polygon of double the number of sides.*



371. Corollary 4. *Tangents at the mid-points of the arcs between adjacent points of contact of the sides of a regular circumscribed polygon form a regular circumscribed polygon of double the number of sides.*



372. Corollary 5. *The perimeter of a regular inscribed polygon is less than that of a regular inscribed polygon of double the number of sides; and the perimeter of a regular circumscribed polygon is greater than that of a regular circumscribed polygon of double the number of sides.*

EXERCISE 59

1. Find the radius of the square whose side is 5 in.

2. Find the side of the square whose radius is 7 in.

3. Find the radius of the equilateral triangle whose side is 2 in.

4. Find the side of the equilateral triangle whose radius is 3 in.

5. Find the apothem of the equilateral triangle whose side is $\sqrt{3}$ in.

6. Find the side of the equilateral triangle whose apothem is $2\sqrt{3}$ in.

7. How many degrees are there in the angle at the center of an equiangular triangle? of a regular hexagon?

8. Given an equilateral triangle inscribed in a circle, to circumscribe an equilateral triangle about the circle.

9. Given an equilateral triangle inscribed in a circle, to inscribe a regular hexagon in the circle, and to circumscribe a regular hexagon about the circle.

10. Given a square inscribed in a circle, to inscribe a regular octagon in the circle, and to circumscribe a regular octagon about the circle.

11. How many degrees are there in the angle at the center of a regular octagon? in each angle of a regular octagon? in the sum of these two angles?

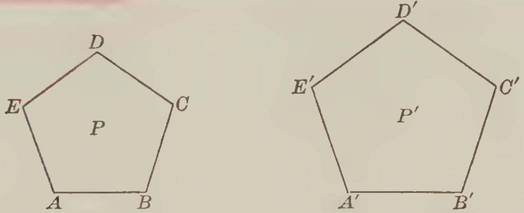
12. What is the area of the square inscribed in a circle of radius 2 in.?

13. The apothem of an equilateral triangle is equal to half the radius.

14. Prove that the apothem of an equilateral triangle is equal to one fourth the diameter of the circumscribed circle. From this show how an equilateral triangle may be inscribed in a circle.

PROPOSITION III. THEOREM

373. *Two regular polygons of the same number of sides are similar.*



Given the regular polygons P and P', each having n sides.

To prove that P *and* P' *are similar.*

Proof. Each angle of either polygon $= \frac{2(n-2)}{n}$ rt. ∠s. § 145

(*Each ∠ of a regular polygon of n sides is equal to* $\frac{2(n-2)}{n}$ *rt. ∠s.*)

Hence the polygons P and P' are mutually equiangular.

Furthermore, $\because AB = BC = CD = DE = EA$,

and $A'B' = B'C' = C'D' = D'E' = E'A'$, § 357

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}. \qquad \text{Ax. 4}$$

Hence the polygons have their corresponding sides proportional and their corresponding angles equal.

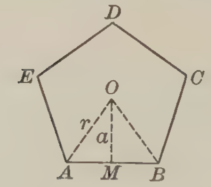
Therefore the two polygons are similar, by § 282. Q.E.D.

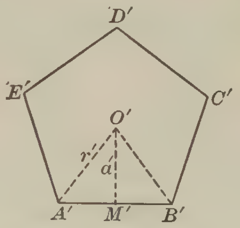
374. COROLLARY. *The areas of two regular polygons of the same number of sides are to each other as the squares on any two corresponding sides.*

For the areas of two similar polygons are to each other as the squares on any two corresponding sides (§ 334), and two regular polygons of the same number of sides are similar.

PROPOSITION IV. THEOREM

375. *The perimeters of two regular polygons of the same number of sides are to each other as their radii, and also as their apothems.*





Given the regular polygons with perimeters p and p', radii r and r', apothems a and a', and centers O and O' respectively.

To prove that $p : p' = r : r' = a : a'$.

Proof. Since the polygons are similar, § 373

$\therefore p : p' = AB : A'B'$. § 291

Furthermore in the isosceles △ OAB and $O'A'B'$,

$\angle O = \angle O'$, § 364

and $OA : OB = O'A' : O'B'$. Ax. 8

(*For each of these ratios equals 1.*)

∴ the △ OAB and $O'A'B'$ are similar. § 288

$\therefore AB : A'B' = r : r'$. § 282

Also △ AMO and $A'M'O'$ are similar. § 286

$\therefore r : r' = a : a'$. § 282

$\therefore p : p' = r : r' = a : a'$, by Ax. 8. Q. E. D.

376. COROLLARY. *The areas of two regular polygons of the same number of sides are to each other as the squares on the radii of the circumscribed circles, and also as the squares on the radii of the inscribed circles.*

EXERCISE 60

1. Find the ratio of the perimeters and the ratio of the areas of two regular hexagons, their sides being 2 in. and 4 in. respectively.

2. Find the ratio of the perimeters and the ratio of the areas of two regular octagons, their sides having the ratio 2 : 6.

3. Find the ratio of the perimeters of two squares whose areas are 121 sq. in. and $30\frac{1}{4}$ sq. in. respectively.

4. Find the ratio of the perimeters and the ratio of the areas of two equilateral triangles whose altitudes are 3 in. and 12 in. respectively.

5. The area of one equiangular triangle is nine times that of another. Required the ratio of their altitudes.

6. The area of the cross section of a steel beam 1 in. thick is 12 sq. in. What is the area of the cross section of a beam of the same proportions and $1\frac{1}{8}$ in. thick?

7. Squares are inscribed in two circles of radii 2 in. and 6 in. respectively. Find the ratio of the areas of the squares, and also the ratio of the perimeters.

8. Squares are inscribed in two circles of radii 2 in. and 8 in. respectively, and on the sides of these squares equilateral triangles are constructed. What is the ratio of the areas of these triangles?

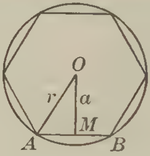
9. A round log a foot in diameter is sawed so as to have the cross section the largest square possible. What is the area of this square? What would be the area of the cross section of the square beam cut from a log of half this diameter?

10. Every equiangular polygon inscribed in a circle is regular if it has an odd number of sides.

11. Every equilateral polygon circumscribed about a circle is regular if it has an odd number of sides.

PROPOSITION V. THEOREM

377. *If the number of sides of a regular inscribed polygon is indefinitely increased, the apothem of the polygon approaches the radius of the circle as its limit.*



Given a regular polygon of n sides inscribed in the circle of radius OA, s being one side and a the apothem.

To prove that a approaches r as a limit, if n is increased indefinitely.

Proof. We know that $a < r$. § 86

Then since $r - a < AM$, § 112

and $AM < s$, § 174

$\therefore r - a < s$. Ax. 10

If n is taken sufficiently great, s, and consequently $r - a$, can be made less than any assigned positive value, however small.

Since $r - a$ can become and remain less than any assigned positive value by increasing n, it follows that

r is the limit of a, by § 204. Q.E.D.

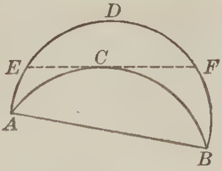
378. COROLLARY. *If the number of sides of a regular inscribed polygon is indefinitely increased, the square on the apothem approaches the square on the radius of the circle as its limit.*

For $r^2 - a^2 = \overline{AM}^2$. § 338

Therefore by taking n sufficiently great, s, and consequently AM, and consequently $r^2 - a^2$, approaches zero as its limit.

PROPOSITION VI. THEOREM

379. *An arc of a circle is less than a line of any kind that envelops it on the convex side and has the same extremities.*



Given BCA an arc of a circle, AB being its chord.

To prove that the arc BCA is less than a line of any kind that envelops this arc and terminates at A and B.

Proof. Of all the lines that can be drawn, each to include the area ABC between itself and the chord AB, there must be at least one shortest line; for all the lines are not equal.

Let BDA be any kind of line enveloping BCA as stated.

The enveloping line BDA cannot be the shortest; for drawing ECF tangent to the arc BCA at any point C, the line $BFCEA < BFDEA$, since $FCE < FDE$. Post. 3

In like manner it can be shown that no other enveloping line can be the shortest.

$\therefore BCA$ is shorter than any enveloping line. Q.E.D.

380. COROLLARY. *A circle is less than the perimeter of any polygon circumscribed about it.*

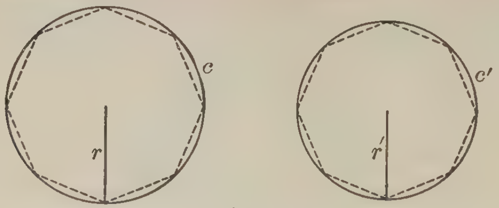
381. Circle as a Limit. From Prop. VI we may now assume:

1. *The circle is the limit which the perimeters of regular inscribed polygons and of similar circumscribed polygons approach, if the number of sides of the polygons is indefinitely increased.*

2. *The area of the circle is the limit which the areas of the inscribed and circumscribed polygons approach.*

PROPOSITION VII. THEOREM

382. *Two circumferences have the same ratio as their radii.*



Given the circles with circumferences c and c', and radii r and r' respectively.

To prove that $c : c' = r : r'$.

Proof. Inscribe in the circles two similar regular polygons, and denote their perimeters by p and p'.

Then $p : p' = r : r'$. § 375

Conceive the number of sides of these regular polygons to be indefinitely increased, the polygons continuing similar.

Then p and p' will have c and c' as limits. § 381

But the ratio $p : p'$ will always be equal to the ratio $r : r'$. § 375

$\therefore pr' = p'r$. § 261

$\therefore cr' = c'r$. § 207

$\therefore c : c' = r : r'$, by § 264. Q.E.D.

383. COROLLARY 1. *The ratio of any circle to its diameter is constant.*

Why does $c : c' = 2r : 2r'$? Then why does $c : 2r = c' : 2r'$?

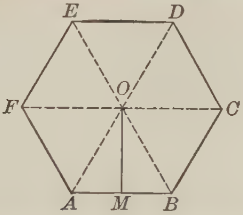
384. The Symbol π. The constant ratio of a circle to its diameter is represented by the Greek letter π (pī).

385. COROLLARY 2. *In any circle $c = 2\pi r$.*

For, by definition, $\pi = \dfrac{c}{2r}$.

PROPOSITION VIII. THEOREM

386. *The area of a regular polygon is equal to half the product of its apothem by its perimeter.*



Given the regular polygon $ABCDEF$, with apothem a, perimeter p, and area s.

To prove that $s = \frac{1}{2}ap$.

Proof. Draw the radii OA, OB, OC, etc., to the successive vertices of the polygon.

The polygon is then divided into as many triangles as it has sides.

The apothem is the common altitude of these △, and the area of each △ is equal to $\frac{1}{2}a$ multiplied by the base. § 325

Hence the area of all the triangles is equal to $\frac{1}{2}a$ multiplied by the sum of all the bases. Ax. 1

But the sum of the areas of all the triangles is equal to the area of the polygon. Ax. 11

And the sum of all the bases of the triangles is equal to the perimeter of the polygon. Ax. 11

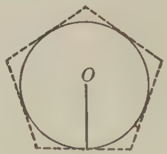
$\therefore s = \frac{1}{2}ap$. Q.E.D.

387. Similar Parts. In different circles *similar arcs, similar sectors,* and *similar segments* are such arcs, sectors, and segments as correspond to equal angles at the center.

For example, two arcs of 30° in different circles are similar arcs, and the corresponding sectors are similar sectors.

PROPOSITION IX. THEOREM

388. *The area of a circle is equal to half the product of its radius by its circumference.*



Given a circle with radius r, circumference c, and area s.

To prove that $s = \frac{1}{2}rc$.

Proof. Circumscribe any regular polygon of n sides, and denote the perimeter of this polygon by p and its area by s'.

Then since r is its apothem, $s' = \frac{1}{2}rp$. § 386

Conceive n to be indefinitely increased.

Then since p approaches c as its limit, § 381

and r is constant,

$\therefore \frac{1}{2}rp$ approaches $\frac{1}{2}rc$ as its limit.

Also s' approaches s as its limit. § 381

But $s' = \frac{1}{2}rp$ always. § 386

$\therefore s = \frac{1}{2}rc$, by § 207. Q. E. D.

389. COROLLARY 1. *The area of a circle is equal to π times the square on its radius.*

For the area of the $\odot = \frac{1}{2}rc = \frac{1}{2}r \times 2\pi r = \pi r^2$.

390. COROLLARY 2. *The areas of two circles are to each other as the squares on their radii.*

391. COROLLARY 3. *The area of a sector is equal to half the product of its radius by its arc.*

For

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{arc of sector}}{\text{circle}}.$$

EXERCISE 61

1. Two circles are constructed with radii $1\frac{1}{32}$ in. and $4\frac{1}{8}$ in. respectively. The circumference of the second is how many times that of the first?

2. The circumference of one circle is three times that of another. The square on the radius of the first is how many times the square on the radius of the second?

3. The circumference of one circle is $2\frac{1}{2}$ times that of another. The equilateral triangle constructed on the diameter of the first has how many times the area of the equilateral triangle constructed on the diameter of the second?

4. A circle with a diameter of 5 in. has a circumference of 15.708 in. What is the circumference of a pipe that has a diameter of 2 in.?

5. A wheel with a circumference of 4 ft. has a diameter of 1.27 ft., expressed to two decimal places. What is the circumference of a wheel with a diameter of $1.58\frac{3}{4}$ ft.?

6. A regular hexagon is 2 in. on a side. Find its apothem and its area to two decimal places.

7. An equilateral triangle is 2 in. on a side. Find its apothem and its area to two decimal places.

8. The radius of one circle is $2\frac{1}{2}$ times that of another. The area of the smaller is 15.2 sq. in. What is the area of the larger?

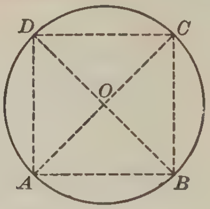
9. The radius of one circle is $3\frac{1}{4}$ times that of another. The area of the smaller is 17.75 sq. in. What is the area of the larger?

10. The circumferences of two cylindrical steel shafts are respectively 3 in. and $1\frac{1}{4}$ in. The area of a cross section of the first is how many times that of a cross section of the second?

11. The arc of a sector of a circle $2\frac{1}{4}$ in. in diameter is $1\frac{3}{4}$ in. What is the area of the sector?

PROPOSITION X. PROBLEM

392. *To inscribe a square in a given circle.*



Given a circle with center O.

Required to inscribe a square in the given circle.

Construction. Draw two diameters AC and BD perpendicular to each other. § 228

Draw AB, BC, CD, and DA.

Then $ABCD$ is the square required. Q.E.F.

Proof. The ∠s CBA, DCB, ADC, BAD are rt. ∠s. § 215

(*An ∠ inscribed in a semicircle is a rt. ∠.*)

The ∠s at the center O being rt. ∠s, Const.

the arcs AB, BC, CD, and DA are equal, § 212

and the sides AB, BC, CD, and DA are equal. § 170

Hence the quadrilateral $ABCD$ is a square, by § 65. Q.E.D.

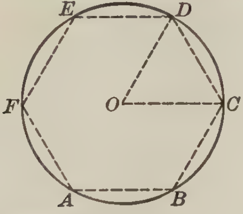
393. COROLLARY. *To inscribe regular polygons of* 8, 16, 32, 64, *etc., sides in a given circle.*

By bisecting the arcs AB, BC, etc., a regular polygon of how many sides may be inscribed in the circle? By continuing the process regular polygons of how many sides may be inscribed?

In general we may say that this corollary allows us to inscribe a regular polygon of 2^n sides, where n is any positive integer. As a special case it is interesting to note that n may equal 1.

PROPOSITION XI. PROBLEM

394. *To inscribe a regular hexagon in a given circle.*



Given a circle with center O.

Required to inscribe a regular hexagon in the given circle.

Construction. From the center O draw any radius, as OC.

With C as a center, and a radius equal to OC, describe an arc intersecting the circle at D.

Draw OD and CD.

Then CD is a side of the regular hexagon required, and therefore the hexagon may be inscribed by applying CD six times as a chord. Q. E. F.

Proof. The $\triangle OCD$ is equiangular. § 75

(*An equilateral triangle is equiangular.*)

Hence the $\angle COD$ is $\frac{1}{3}$ of 2 rt. $\angle$s, or $\frac{1}{6}$ of 4 rt. $\angle$s. § 107

$\therefore$ the arc CD is $\frac{1}{6}$ of the circle.

$\therefore$ the chord CD is a side of a regular inscribed hexagon. Q. E. D.

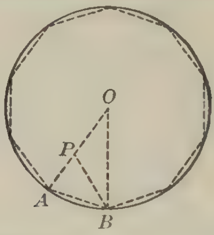
395. COROLLARY 1. *To inscribe an equilateral triangle in a given circle.*

By joining the alternate vertices of a regular inscribed hexagon, an equilateral triangle may be inscribed.

396. COROLLARY 2. *To inscribe regular polygons of* 12, 24, 48, *etc., sides in a given circle.*

PROPOSITION XII. PROBLEM

397. *To inscribe a regular decagon in a given circle.*



Given a circle with center *O*.

Required to inscribe a regular decagon in the given circle.

Construction. Draw any radius OA,

and divide it in extreme and mean ratio, § 311

so that $OA : OP = OP : AP$.

From A as a center, with a radius equal to OP,

describe an arc intersecting the circle at B.

Draw AB.

Then AB is a side of the regular decagon required, and therefore the regular decagon may be inscribed by applying AB ten times as a chord. Q. E. F.

Proof. Draw PB and OB.

Then $OA : OP = OP : AP$, Const.

and $AB = OP$. Const.

$\therefore OA : AB = AB : AP$. Ax. 9

Moreover, $\angle BAO = \angle BAP$. Iden.

Hence the $\triangle$ OAB and BAP are similar. § 288

But the $\triangle OAB$ is isosceles. § 162

$\therefore \triangle BAP$, which is similar to $\triangle OAB$, is isosceles, § 282

and $AB = BP = OP$. § 62

The $\triangle PBO$ being isosceles, the $\angle O = \angle OBP$. § 74

But the $\angle APB = \angle O + \angle OBP = 2\angle O$. § 111

Hence $\angle BAP = 2\angle O$,

and $\angle OBA = 2\angle O$. Ax. 9

$\therefore$ the sum of the $\angle$s of the $\triangle OAB = 5\angle O = 2$ rt. $\angle$s, § 107

and $\angle O = \frac{1}{5}$ of 2 rt. $\angle$s, or $\frac{1}{10}$ of 4 rt. $\angle$s. Ax. 4

Therefore the arc AB is $\frac{1}{10}$ of the circle. § 212

$\therefore$ the chord AB is a side of a regular inscribed decagon. Q. E. D.

398. COROLLARY 1. *To inscribe a regular pentagon in a given circle.*

399. COROLLARY 2. *To inscribe regular polygons of* 20, 40, 80, *etc., sides in a given circle.*

By bisecting the arcs subtended by the sides of a regular inscribed decagon a regular polygon of how many sides may be inscribed? By continuing the process regular polygons of how many sides may be inscribed?

EXERCISE 62

If r denotes the radius of a regular inscribed polygon, a the apothem, s one side, A an angle, and C the angle at the center, show that:

1. In a regular inscribed triangle $s = r\sqrt{3}$, $a = \frac{1}{2}r$, $A = 60°$, $C = 120°$.

2. In a regular inscribed quadrilateral $s = r\sqrt{2}$, $a = \frac{1}{2}r\sqrt{2}$, $A = 90°$, $C = 90°$.

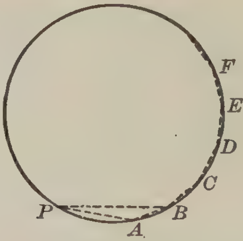
3. In a regular inscribed hexagon $s = r$, $a = \frac{1}{2}r\sqrt{3}$, $A = 120°$, $C = 60°$.

4. In a regular inscribed decagon

$$s = \frac{r(\sqrt{5}-1)}{2},\ a = \frac{1}{4}r\sqrt{10+2\sqrt{5}},\ A = 144°,\ C = 36°.$$

PROPOSITION XIII. PROBLEM

400. *To inscribe in a given circle a regular pentadecagon, or polygon of fifteen sides.*



Given a circle.

Required to inscribe a regular pentadecagon in the given circle.

Construction. Draw a chord PB equal to the radius of the circle, a chord PA equal to a side of the regular inscribed decagon, and draw AB.

Then AB is a side of the regular pentadecagon required, and therefore the regular pentadecagon may be inscribed by applying AB fifteen times as a chord. Q. E. F.

Proof. The arc PB is $\frac{1}{6}$ of the circle, § 394

and the arc PA is $\frac{1}{10}$ of the circle. § 397

Hence the arc AB is $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$, of the circle. Ax. 2

Therefore the chord AB is a side of the regular inscribed pentadecagon required. Q. E. D.

401. COROLLARY. *To inscribe regular polygons of* 30, 60, 120, *etc., sides in a given circle.*

By bisecting the arcs AB, BC, etc., a regular polygon of how many sides may be inscribed? By continuing the process regular polygons of how many sides may be inscribed? In general we may say that a regular polygon of $15 \cdot 2^n$ sides may be inscribed in this manner.

EXERCISE 63

1. A five-cent piece is placed on the table. How many five-cent pieces can be placed around it, each tangent to it and tangent to two of the others? Prove it.

2. What is the perimeter of an equilateral triangle inscribed in a circle with radius 1 in.?

3. What is the perimeter of an equilateral triangle circumscribed about a circle with radius 1 in.?

4. What is the perimeter of a regular hexagon circumscribed about a circle with radius 1 in.?

Required to circumscribe about a given circle the following regular polygons:

5. Triangle.

6. Quadrilateral.

7. Hexagon.

8. Octagon.

9. Pentagon.

10. Decagon.

11. Required to describe a circle whose circumference equals the sum of the circumferences of two circles of given radii.

12. Required to describe a circle whose area equals the sum of the areas of two circles of given radii.

13. Required to describe a circle having three times the area of a given circle.

Required to construct an angle of:

14. 18°. **15.** 36°. **16.** 9°. **17.** 12°. **18.** 24°.

Required to construct with a side of given length:

19. An equilateral triangle.

20. A square.

21. A regular hexagon.

22. A regular octagon.

23. A regular pentagon.

24. A regular decagon.

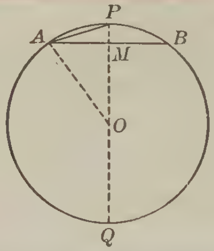
25. A regular dodecagon.

26. A regular pentadecagon.

27. From a circular log 16 in. in diameter a builder wishes to cut a column with its cross section as large a regular octagon as possible. Find the length of each side.

PROPOSITION XIV. PROBLEM

402. *Given the side and the radius of a regular inscribed polygon, to find the side of the regular inscribed polygon of double the number of sides.*



Given AB, the side of a regular inscribed polygon of radius OA.

Required to find AP, a side of the regular inscribed polygon of double the number of sides.

Solution. Denote the radius by r, and the side AB by s.

Draw the diameter $PQ \perp$ to AB, and draw OA.

Then $AM = \frac{1}{2} s$. § 174

In the rt. $\triangle AOM$, $\overline{OM}^2 = r^2 - \frac{1}{4} s^2$. § 338

Therefore $OM = \sqrt{r^2 - \frac{1}{4} s^2}$. Ax. 5

Since $PM + OM = r$, Ax. 11

therefore $PM = r - OM$ Ax. 2

$= r - \sqrt{r^2 - \frac{1}{4} s^2}$. Ax. 9

Furthermore $\overline{AP}^2 = PQ \times PM$. § 298

But $PQ = 2r$, and $PM = r - \sqrt{r^2 - \frac{1}{4} s^2}$.

$\therefore \overline{AP}^2 = 2r\left(r - \sqrt{r^2 - \frac{1}{4} s^2}\right)$. Ax. 9

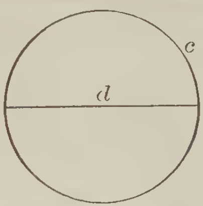
$\therefore AP = \sqrt{2r\left(r - \sqrt{r^2 - \frac{1}{4} s^2}\right)}$ Ax. 5

$= \sqrt{r\left(2r - \sqrt{4r^2 - s^2}\right)}$. Q.E.F

403. COROLLARY. *If* $r = 1$, $AP = \sqrt{2 - \sqrt{4 - s^2}}$.

Proposition XV. Problem

404. *To find the numerical value of the ratio of the circumference of a circle to its diameter.*



Given a circle of circumference *c* and diameter *d*.

Required to find the numerical value of $\frac{c}{d}$ *or* π.

Solution. By § 385, $2\pi r = c$. $\therefore \pi = \frac{1}{2}c$ when $r = 1$.

Let s_6 (read "*s* sub six") be the length of a side of a regular polygon of 6 sides, s_{12} of 12 sides, and so on.

If $r = 1$, by § 394, $s_6 = 1$, and by § 403 we have

Form of Computation	Length of Side	Length of Perimeter
$s_{12} = \sqrt{2 - \sqrt{4 - 1^2}}$	0.51763809	6.21165708
$s_{24} = \sqrt{2 - \sqrt{4 - (0.51763809)^2}}$	0.26105238	6.26525722
$s_{48} = \sqrt{2 - \sqrt{4 - (0.26105238)^2}}$	0.13080626	6.27870041
$s_{96} = \sqrt{2 - \sqrt{4 - (0.13080626)^2}}$	0.06543817	6.28206396
$s_{192} = \sqrt{2 - \sqrt{4 - (0.06543817)^2}}$	0.03272346	6.28290510
$s_{384} = \sqrt{2 - \sqrt{4 - (0.03272346)^2}}$	0.01636228	6.28311544
$s_{768} = \sqrt{2 - \sqrt{4 - (0.01636228)^2}}$	0.00818121	6.28316941

$\therefore c = 6.28317$ nearly; that is, $\pi = 3.14159$ nearly. Q.E.F.

π is an incommensurable number. We generally take

$$\pi = 3.1416, \text{ or } 3\tfrac{1}{7}, \text{ and } \frac{1}{\pi} = 0.31831.$$

EXERCISE 64

Problems of Computation

Using the value 3.1416 for π, find the circumferences of circles with radii as follows:

1. 3 in.
2. 5 in.
3. 2.7 in.
4. 3.4 in.
5. $7\frac{1}{2}$ in.
6. $6\frac{3}{4}$ in.
7. 2 ft. 8 in.
8. 3 ft. 7 in.

Find the circumferences of circles with diameters as follows:

9. 9 in.
10. 12 in.
11. 5.9 in.
12. 7.3 in.
13. $2\frac{1}{2}$ ft.
14. $3\frac{1}{8}$ in.
15. 29 centimeters.
16. 47 millimeters.

Find the radii of circles with circumferences as follows:

17. 7π.
18. $3\frac{1}{3}\pi$.
19. 15.708 in.
20. 21.9912 in.
21. 18.8496 in.
22. 125.664 in.
23. 345.576 ft.
24. 3487.176 in.

Find the diameters of circles with circumferences as follows:

25. 15π.
26. π^2.
27. $2\pi r$.
28. $7\pi a^2$.
29. 188.496 in.
30. 219.912 in.
31. 3361.512 in.
32. 3173.016 in.

Find the areas of circles with radii as follows:

33. $5x$.
34. 2π.
35. 27 ft.
36. 4.8 ft.
37. $3\frac{1}{8}$ in.
38. $4\frac{5}{8}$ in.
39. 2 ft. 6 in.
40. 7 ft. 9 in

Find the areas of circles with diameters as follows:

41. $16ab$.
42. $24\pi^2$.
43. 2.5 ft.
44. 7.3 in.
45. $3\frac{2}{3}$ yd.
46. $4\frac{5}{8}$ yd.
47. 3 ft. 2 in.
48. 4 ft. 1 in.

Find the areas of circles with circumferences as follows:

49. 2π.
50. 4π.
51. πa.
52. $14\pi a^2$.
53. 18.8496 in.
54. 329.868 in.
55. 333.0096 in.
56. 364.4256 in.

Find the radii of circles with areas as follows:

57. $\pi a^2 b^4$.
58. $4\pi m^4 n^6$.
59. π.
60. 2π.
61. 12.5664.
62. 28.2744.
63. 78.54.
64. 113.0976.

EXERCISE 65

PROBLEMS OF CONSTRUCTION

1. To inscribe in a given circle a regular polygon similar to a given regular polygon.

2. To divide by a concentric circle the area of a given circle into two equivalent parts.

3. To divide by concentric circles the area of a given circle into n equivalent parts.

4. To describe a circle whose circumference is equal to the difference of two circumferences of given radii.

5. To describe a circle the ratio of whose area to that of a given circle shall be equal to the given ratio $m:n$.

6. To construct a regular pentagon, given one of the diagonals.

7. To draw a tangent to a given circle such that the segment intercepted between the point of contact and a given line shall have a given length.

8. In a given equilateral triangle to inscribe three equal circles tangent each to the other two, each circle being tangent to two sides of the triangle.

9. In a given square to inscribe four equal circles, so that each circle shall be tangent to two of the others and also tangent to two sides of the square.

10. In a given square to inscribe four equal circles, so that each circle shall be tangent to two of the others and also tangent to one side and only one side of the square.

11. To draw a common secant to two given circles exterior to each other, such that the intercepted chords shall have the given lengths a and b.

12. To draw through a point of intersection of two given intersecting circles a common secant of a given length.

EXERCISE 66

Problems of Loci

1. Find the locus of the center of the circle inscribed in a triangle that has a given base and a given angle at the vertex.

2. Find the locus of the intersection of the perpendiculars from the three vertices to the opposite sides of a triangle that has a given base and a given angle at the vertex.

3. Find the locus of the extremity of a tangent to a given circle, if the length of the tangent is equal to a given line.

4. Find the locus of a point from which tangents drawn to a given circle form a given angle.

5. Find the locus of the mid-point of a line drawn from a given point to a given line.

6. Find the locus of the vertex of a triangle that has a given base and a given altitude.

7. Find the locus of a point the sum of whose distances from two given parallel lines is constant.

8. Find the locus of a point the difference of whose distances from two given parallel lines is constant.

9. Find the locus of a point the sum of whose distances from two given intersecting lines is constant.

10. Find the locus of a point the difference of whose distances from two given intersecting lines is constant.

11. Find the locus of a point whose distances from two given points are in the ratio $m : n$.

12. Find the locus of a point whose distances from two given parallel lines are in the ratio $m : n$.

13. Find the locus of a point whose distances from two given intersecting lines are in the ratio $m : n$.

14. Find the locus of a point the sum of the squares of whose distances from two given points is constant.

EXERCISE 67

EXAMINATION QUESTIONS

1. Each side of a triangle is $2n$ centimeters, and about each vertex as a center, with a radius of n centimeters, a circle is described. Find the area bounded by the three arcs that lie outside the triangle, and the area bounded by the three arcs that lie within the triangle.

2. Upon a line AB a segment of a circle containing 240° is constructed, and in the segment any chord CD subtending an arc of 60° is drawn. Find the locus of the intersection of AC and BD, and also of the intersection of AD and BC.

3. Three successive vertices of a regular octagon are A, B, and C. If the length AB is a, compute the length AC.

4. The areas of similar segments of circles are proportional to the squares on their radii.

5. An arc of a certain circle is 100 ft. long and subtends an angle of 25° at the center. Compute the radius of the circle correct to one decimal place.

6. Given a circle whose radius is 16, find the perimeter and the area of the regular inscribed octagon.

7. If two circles intersect at the points A and B, and through A a variable secant is drawn, cutting the circles in C and D, the angle CBD is constant for all positions of the secant.

8. If A and B are two fixed points on a given circle, and P and Q are the extremities of a variable diameter of the same circle, find the locus of the point of intersection of the lines AP and BQ.

9. The radius of a circle is 10 ft. Two parallel chords are drawn, each equal to the radius. Find that part of the area of the circle lying between the parallel chords.

The propositions in Exercise 67 are taken from recent college entrance examination papers.

EXERCISE 68

Formulas

If r denotes the radius of a circle, and s one side of a regular inscribed polygon, prove the following, and find the value of s to two decimal places when $r = 1$:

1. In an equilateral triangle $s = r\sqrt{3}$.

2. In a square $s = r\sqrt{2}$.

3. In a regular pentagon $s = \frac{1}{2}r\sqrt{10 - 2\sqrt{5}}$.

4. In a regular hexagon $s = r$.

5. In a regular octagon $s = r\sqrt{2 - \sqrt{2}}$.

6. In a regular decagon $s = \frac{1}{2}r(\sqrt{5} - 1)$.

7. In a regular dodecagon $s = r\sqrt{2 - \sqrt{3}}$.

8. A regular pentagon is inscribed in a circle whose radius is r. If the side is s, find the apothem.

9. A regular polygon is inscribed in a circle whose radius is r. If the side is s, show that the apothem is $\frac{1}{2}\sqrt{4r^2 - s^2}$.

10. If the radius of a circle is r, and the side of an inscribed regular polygon is s, show that the side of the similar circumscribed regular polygon is $\dfrac{2sr}{\sqrt{4r^2 - s^2}}$.

11. Three equal circles are described, each tangent to the other two. If the common radius is r, find the area contained between the circles.

12. Given p, P, the perimeters of regular polygons of n sides inscribed in and circumscribed about a given circle. Find p', P', the perimeters of regular polygons of $2n$ sides inscribed in and circumscribed about the given circle.

13. A circular plot of land d ft. in diameter is surrounded by a walk w ft. wide. Find the area of the circular plot and the area of the walk.

EXERCISE 69

Applied Problems

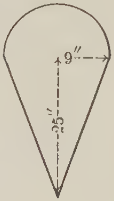
1. The diameter of a bicycle wheel is 28 in. How many revolutions does the wheel make in going 10 mi.?

2. Find the diameter of a carriage wheel that makes 264 revolutions in going half a mile.

3. A circular pond 100 yd. in diameter is surrounded by a walk 10 ft. wide. Find the area of the walk.

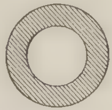
4. The span (chord) of a bridge in the form of a circular arc is 120 ft., and the highest point of the arch is 15 ft. above the piers. Find the radius of the arc.

5. Two branch water pipes lead into a main pipe. It is necessary that the cross-section area of the main pipe shall equal the sum of the cross sections of the two branch pipes. The diameters of the branch pipes are respectively 3 in. and 4 in. Required the diameter of the main pipe.



6. A kite is made as here shown, the semicircle having a radius of 9 in., and the triangle a height of 25 in. Find the area of the kite.

7. In making a drawing for an arch it is required to mark off on a circle drawn with a radius of 5 in. an arc that shall be 8 in. long. This is best done by finding the angle at the center. How many degrees are there in this angle?



8. In an iron washer here shown, the diameter of the hole is $1\frac{3}{8}$ in. and the width of the washer is $\frac{3}{8}$ in. Find the area of one face of the washer.

9. Find the area of a fan that opens out into a sector of 120°, the radius being $9\frac{3}{8}$ in.

10. The area of a fan that opens out into a sector of 111° is 96.866 sq. in. What is the radius? (Take $\pi = 3.1416$.)

EXERCISE 70

REVIEW QUESTIONS

1. What is meant by a regular polygon? by its radius? by its center? by its apothem?

2. What other names are there for a regular triangle and a regular quadrilateral?

3. If one angle of a regular polygon is known, how can the number of sides be determined?

4. The sides of two regular polygons of n sides are respectively s and s'. What is the ratio of their radii? of their apothems? of their perimeters? of their areas?

5. The diameters of two circles are d and d' respectively. What is the ratio of their radii? of their circumferences? of their areas?

6. If the number of sides of a regular inscribed polygon is indefinitely increased, what is the limit of the apothem? of each side? of the perimeter? of the area? of the angle at the center? of each angle of the polygon?

7. How do you find the area of a regular polygon? of an irregular polygon? of a square? of a triangle? of a parallelogram? of a circle? of a trapezoid? of a sector?

8. What regular polygons have you learned to inscribe in a circle? Name three regular polygons that you have not learned to inscribe.

9. Given the circumference of a circle, how can the area of the circle be found?

10. Given the area of a circle, how can the circumference of the circle be found?

11. What is the radius of the circle of which the number of linear units of circumference is equal to the number of square units of area?

EXERCISE 71

GENERAL REVIEW OF PLANE GEOMETRY

Write a classification of the different kinds of:

1. Lines.
2. Angles.
3. Triangles.
4. Quadrilaterals.
5. Polygons.
6. Parallelograms.

State the conditions under which:

7. Two triangles are congruent.

8. Two parallelograms are congruent.

9. Two triangles are similar.

10. Two straight lines are parallel.

11. Two parallelograms are equivalent.

12. Two polygons are similar.

Complete the following statements in the most general manner:

13. In any triangle the square on the side opposite $\cdots$.

14. If two parallel lines are cut by a transversal, $\cdots$.

15. If four quantities are in proportion, they are in proportion by $\cdots$.

16. If two secants of a circle intersect, the angle formed is measured by $\cdots$.

17. The perimeters of two similar polygons are to each other as $\cdots$.

18. The areas of two similar polygons are to each other as $\cdots$.

19. The area of a circle is equal to $\cdots$.

20. In the same circle or in equal circles equal chords $\cdots$.

21. In the same circle or in equal circles the central angles subtended by two arcs are $\cdots$.

22. If two secants of a circle intersect within, on, or outside the circle, the product of $\cdots$.

23. If four lines meet in a point so that the opposite angles are equal, these lines form two intersecting straight lines.

24. If squares are constructed outwardly on the six sides of a regular hexagon, the exterior vertices of these squares are the vertices of a regular dodecagon.

25. In a right triangle the line joining the vertex of the right angle to the mid-point of the hypotenuse is equal to half the hypotenuse.

26. No two lines drawn from the vertices of the base angles of a triangle to the opposite sides can bisect each other.

27. The rhombus is the only parallelogram that can be circumscribed about a circle.

28. The square is the only rectangle that can be circumscribed about a circle.

29. No oblique parallelogram can be inscribed in a circle.

30. If two triangles have equal bases and equal vertical angles, the two circumscribing circles have equal diameters.

31. If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.

32. If the three points of contact of a circle inscribed in a triangle are joined, the angles of the resulting triangle are all acute.

33. The diagonals of a regular pentagon intersect at the vertices of another regular pentagon.

34. If two perpendicular radii of a circle are produced to intersect a tangent to the circle, the other tangents from the two points of intersection are parallel.

35. The line that joins the feet of the perpendiculars drawn from the extremities of the base of an isosceles triangle to the equal sides is parallel to the base.

36. The sum of the perpendiculars drawn to the sides of a regular polygon from any point within the polygon is equal to the apothem multiplied by the number of sides.

37. If two consecutive angles of a quadrilateral are right angles, the bisectors of the other two angles are perpendicular.

38. If two opposite angles of a quadrilateral are right angles, the bisectors of the other two angles are parallel.

39. The two lines that join the mid-points of opposite sides of a quadrilateral bisect each other.

40. The sum of the angles at the vertices of a five-pointed star is equal to two right angles.

41. The segments of any line intercepted between two concentric circles are equal.

42. The diagonals of a trapezoid divide each other into segments which are proportional.

43. Given the mid-points of the sides of a triangle, to construct the triangle.

44. To divide a given triangle into two equivalent parts by a line perpendicular to the base.

45. To draw a tangent to a given circle that shall also be perpendicular to a given line.

46. To divide a given line into two segments such that the square on one shall be double the square on the other.

47. If any two consecutive sides of an inscribed hexagon are respectively parallel to their opposite sides, the remaining two sides are parallel.

48. If through any given point in the common chord of two intersecting circles two other chords are drawn, one in each circle, their four extremities will all lie on a third circle.

49. If two chords intersect at right angles within a circle, the sum of the squares on their segments equals the square on the diameter. Investigate the case in which the chords intersect outside the circle; also the case in which they intersect on the circle.

50. The lines bisecting any angle of an inscribed quadrilateral and the opposite exterior angle intersect on the circle.

51. The sum of the perpendiculars from any point in an equilateral triangle to the three sides is constant.

52. The perpendiculars from the vertices of a triangle upon the opposite sides cut one another into segments that are reciprocally proportional to each other.

53. The area of a triangle is equal to half the product of its perimeter by the radius of the inscribed circle.

54. The perimeter of a triangle is to one side as the perpendicular from the opposite vertex is to the radius of the inscribed circle.

55. The area of a square inscribed in a semicircle is equal to two fifths of the area of the square inscribed in the circle.

56. The diagonals of any inscribed quadrilateral divide it into two pairs of similar triangles.

57. To draw a line whose length is $\sqrt{7\frac{1}{2}}$ in.

58. If two equivalent triangles are on the same base and the same side of the base, any line cutting the triangles, and parallel to the base, cuts off equal areas from the triangles.

59. To divide a given arc of a circle into two parts such that their chords shall be in a given ratio.

60. The area between two concentric circles may be found by multiplying half the sum of the two circumferences by the difference between the radii.

61. Find the length of the belt connecting two wheels of the same size, if the radius of each wheel is 18 in., the distance between the centers 6 ft., and 4 in. is allowed for sagging.

62. To construct a regular inscribed heptagon draftsmen sometimes use for a side half the side of an inscribed equilateral triangle. Construct such a figure with the compasses, and state whether the rule seems exact or only approximate.

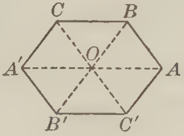
APPENDIX

405. Subjects Treated. Of the many additional subjects that may occupy the attention of the student of plane geometry if time permits, two are of special interest. These are Symmetry, and Maxima and Minima.

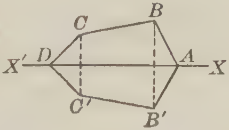
406. Symmetric Points. Two points are said to be *symmetric with respect to a point*, called the *center of symmetry*, if this third point bisects the straight line which joins the two points.

Two points are said to be *symmetric with respect to an axis*, if a straight line, called the *axis of symmetry*, is the perpendicular bisector of the line joining them.

407. Symmetric Figure. A figure is said to be *symmetric with respect to a point*, if the point bisects every straight line drawn through it and terminated by the boundary of the figure.

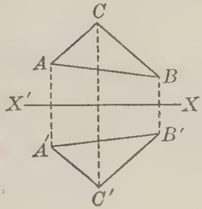


A figure is said to be *symmetric with respect to an axis*, if the axis bisects every perpendicular through it and terminated by the boundary of the figure.



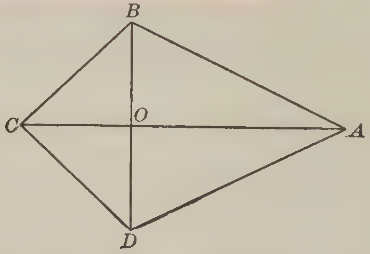
Evidently this will be the case if one part coincides with another part when folded over the axis.

408. Two Symmetric Figures. Two figures are said to be *symmetric with respect to a point* or *symmetric with respect to an axis*, if every point of each has a corresponding symmetric point in the other.



Proposition I. Theorem

409. *A quadrilateral that has two adjacent sides equal, and the other two sides equal, is symmetric with respect to the diagonal joining the vertices of the angles formed by the equal sides; and the diagonals are perpendicular to each other.*



Given the quadrilateral *ABCD*, having *AB* equal to *AD*, and *CB* equal to *CD*, and having the diagonals *AC* and *BD*.

To prove that the diagonal AC is an axis of symmetry, and that AC is ⊥ to BD.

Proof. In the $\triangle$ ABC and ADC,

$AB = AD$, and $CB = CD$, Given

and $AC = AC$. Iden.

$\therefore \triangle ABC$ is congruent to $\triangle ADC$. § 80

$\therefore \angle BAC = \angle CAD$, and $\angle ACB = \angle DCA$. § 67

Hence, if $\triangle ABC$ is turned on AC as an axis until it falls on $\triangle ADC$, AB will fall on AD, CB on CD, and OB on OD.

$\therefore$ the $\triangle ABC$ will coincide with the $\triangle ADC$.

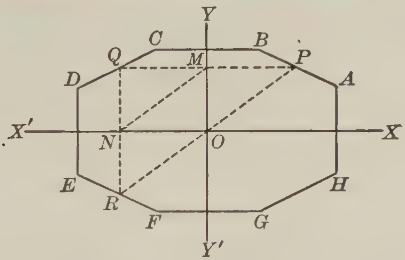
$\therefore AC$ will bisect every perpendicular drawn through it and terminated by the boundary of the figure.

$\therefore AC$ is an axis of symmetry. § 407

$\therefore AC$ is $\perp$ to BD, by § 406. Q. E. D.

PROPOSITION II. THEOREM

410. *If a figure is symmetric with respect to two axes perpendicular to each other, it is symmetric with respect to their intersection as a center.*



Given the figure $ABCDEFGH$, symmetric with respect to the two perpendicular axes XX', YY', which intersect at O.

To prove that O is the center of symmetry of the figure.

Proof. Let P be any point in the perimeter.

Draw $PMQ \perp$ to YY', and $QNR \perp$ to XX'. § 227

Then PQ is $\parallel$ to XX', and QR is $\parallel$ to YY'. § 95

Draw PO, OR, and MN.

Then $QN = NR$. § 407

(*The figure is given as symmetric with respect to XX'.*)

But $QN = MO$. § 127

$\therefore NR = MO$. Ax. 8

$\therefore RO$ is equal and parallel to NM. § 130

In like manner, OP is equal and parallel to NM.

$\therefore ROP$ is a straight line. § 94

$\therefore O$ bisects PR, any straight line, and hence bisects every straight line drawn through O and terminated by the perimeter.

$\therefore O$ is the center of symmetry of the figure, by § 407. Q.E.D.

EXERCISE 72

1. Draw a figure showing the number of axes of symmetry possessed by a square.

2. Draw a figure showing the number of axes of symmetry possessed by a regular hexagon.

3. Draw a figure showing six of the unlimited number of axes of symmetry of a circle, and showing the center of symmetry.

4. Show by drawings that two congruent triangles may be placed in a position of symmetry with respect to an axis. In one of the drawings let a common side be the axis.

5. Show by a drawing that two congruent triangles may be placed in a position of symmetry with respect to a center.

6. Two figures symmetric with respect to an axis are congruent.

7. Two figures symmetric with respect to a center are congruent.

8. Make a list of quadrilaterals that are symmetric with respect to an axis.

9. Make a list of quadrilaterals that are symmetric with respect to a center.

10. What kinds of regular polygons are symmetric with respect both to a center and to an axis? Prove this for the hexagon.

11. A circle is symmetric with respect to its center as a center of symmetry, and is also symmetric with respect to any diameter as an axis.

12. An isosceles triangle is symmetric with respect to an axis, and therefore the angles opposite the equal sides are equal.

13. Two tangents drawn to a circle from the same point are symmetric with respect to an axis.

14. The four common tangents to two given circles form, together with the circles, a figure symmetric with respect to the line of centers as an axis.

411. Maxima and Minima. Among geometric magnitudes that satisfy given conditions, the *greatest* is called the *maximum*, and the *smallest* is called the *minimum.*

The plural of maximum is *maxima*, and the plural of minimum is *minima.*

Among geometric magnitudes that satisfy given conditions, there may be several equal magnitudes that are greater than any others. In this case all are called maxima.

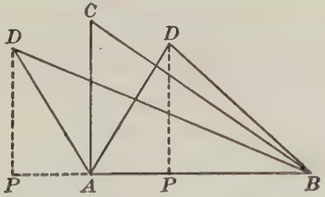
Similarly there may be several minima magnitudes of a given kind.

412. Isoperimetric Polygons. Polygons which have equal perimeters are called *isoperimetric polygons.*

If the circumference of a circle equals the perimeter of a polygon, the circle and the polygon are said to be isoperimetric, and similarly for all other closed figures in a plane.

PROPOSITION III. THEOREM

413. *Of all triangles having two given sides, that in which these sides include a right angle is the maximum.*



Given the triangles ABC and ABD, with AB and CA equal to AB and DA respectively, and with angle BAC a right angle.

To prove that $\triangle ABC > \triangle ABD$.

Proof. From D draw the altitude DP. § 227

Then $DA > DP$. § 86

But $DA = CA$. Given

$\therefore CA > DP$. Ax. 9

$\therefore \triangle ABC > \triangle ABD$, by § 327. Q.E.D.

PROPOSITION IV. THEOREM

414. *Of all isoperimetric triangles having the same base the isosceles triangle is the maximum.*

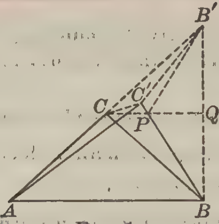


FIG. 1

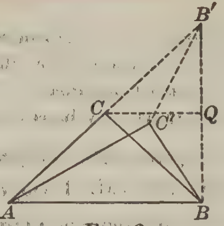


FIG. 2

Given the triangles ABC and ABC' having equal perimeters, and having AC equal to BC, and AC' not equal to BC'.

To prove that $\triangle ABC > \triangle ABC'$.

Proof. Produce AC to B', making $CB' = AC$.

Draw BB' and $C'B'$, and draw $CQ \parallel$ to AB.

Then since $AC = CB'$, $\therefore BQ = QB'$. § 135

And since $CA = CB = CB'$, $\therefore \angle B'BA$ is a rt. $\angle$. § 215

$\therefore CQ$ is $\perp$ to BB'. § 97

C' cannot lie on AB', for if it could, then $CC' + C'B$ would equal CB, which is impossible. Post. 1

Then since $AC + CB' < AC' + C'B'$, § 112

$\therefore AC + CB < AC' + C'B'$. Ax. 9

$\therefore AC' + C'B < AC' + C'B'$. Ax. 9

$\therefore C'B < C'B'$. Ax. 6

$\therefore C'$ cannot lie on CQ, for then $C'B$ would equal $C'B'$. § 150

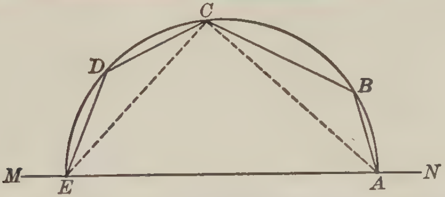
C' cannot lie above CQ (Fig. 1), for $C'B'$, which $< C'P + PB'$, would be less than $C'B$, which equals $C'P + PB'$.

$\therefore C'$ must lie below CQ, as in Fig. 2.

$\therefore \triangle ABC > \triangle ABC'$, by § 327. Q. E. D.

PROPOSITION V. THEOREM

415. *Of all polygons with sides all given but one, the maximum can be inscribed in the semicircle which has the undetermined side for its diameter.*



Given $ABCDE$, **the maximum of polygons with sides** AB, BC, CD, DE, **having the vertices** A **and** E **on the line** MN.

To prove that $ABCDE$ *can be inscribed in the semicircle having* EA *for its diameter.*

Proof. From any vertex, as C, draw CA and CE.

The $\triangle ACE$ must be the maximum of all ▵ having the sides CA and CE, and the third side on MN; otherwise, by increasing or diminishing the $\angle ECA$, keeping the lengths of the sides CA and CE unchanged, but sliding the extremities A and E along the line MN, we could increase the $\triangle ACE$, while the rest of the polygon would remain unchanged; and therefore we could increase the polygon. But this is contrary to the hypothesis that the polygon is the maximum polygon.

Hence the $\triangle ACE$ is the maximum of triangles that have the sides CA and CE.

Therefore the $\angle ACE$ is a right angle. § 413

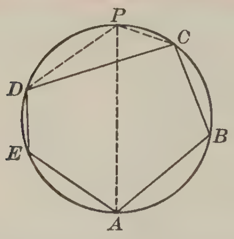
Therefore C lies on the semicircle having EA for its diameter. § 215

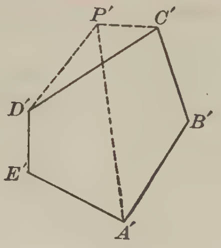
Hence every vertex lies on this semicircle.

That is, the maximum polygon can be inscribed in the semicircle having the undetermined side for its diameter. Q. E. D.

Proposition VI. Theorem

416. *Of all polygons with given sides, one that can be inscribed in a circle is the maximum.*





Given the polygon $ABCDE$ inscribed in a circle, and the polygon $A'B'C'D'E'$ which has its sides equal respectively to the sides of $ABCDE$, but which cannot be inscribed in a circle.

To prove that $ABCDE > A'B'C'D'E'$.

Proof. Draw the diameter AP, and draw CP and PD.

Upon $C'D'$ as a base, construct the $\triangle C'P'D'$ congruent to the $\triangle CPD$, and draw $A'P'$.

Since, by hypothesis, a $\odot$ cannot pass through all the vertices of $A'B'C'P'D'E'$, one or both of the parts $A'P'D'E'$, $A'B'C'P'$ cannot be inscribed in a semicircle.

Neither $A'P'D'E'$ or $A'B'C'P'$ can be greater than its corresponding part. § 415

(*Of all polygons with sides all given but one, the maximum can be inscribed in the semicircle which has the undetermined side for its diameter.*)

Therefore one of the parts $A'P'D'E'$, $A'B'C'P'$ must be less than, and the other cannot be greater than, the corresponding part of $ABCPDE$.

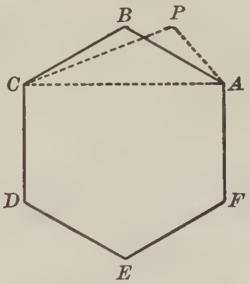
$$\therefore ABCPDE > A'B'C'P'D'E'.$$

Take from the two figures the congruent $\triangle$s CPD and $C'P'D'$.

Then $ABCDE > A'B'C'D'E'$. by Ax. 6. Q.E.D.

PROPOSITION VII. THEOREM

417. *Of isoperimetric polygons of a given number of sides, the maximum is equilateral.*



Given the polygon *ABCDEF*, the maximum of isoperimetric polygons of *n* sides.

To prove that the polygon ABCDEF is equilateral.

Proof. Draw AC.

The $\triangle ABC$ must be the maximum of all the $\triangle$s which are formed upon AC with a perimeter equal to that of $\triangle ABC$.

Otherwise a greater $\triangle APC$ could be substituted for $\triangle ABC$, without changing the perimeter of the polygon.

But this is inconsistent with the fact that the polygon $ABCDEF$ is given as the maximum polygon.

$\therefore$ the $\triangle ABC$ is isosceles. § 414

$\therefore AB = BC.$

Similarly $BC = CD$, $CD = DE$, and so on.

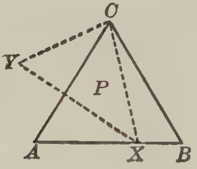
$\therefore$ the polygon $ABCDEF$ is equilateral. Q.E.D.

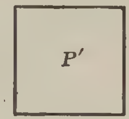
418. COROLLARY. *The maximum of isoperimetric polygons of a given number of sides is a regular polygon.*

For the maximum polygon is equilateral (§ 417), and can be inscribed in a circle (§ 416). Therefore the maximum polygon is regular (§ 365).

PROPOSITION VIII. THEOREM

419. *Of isoperimetric regular polygons, that which has the greatest number of sides is the maximum.*





Given the regular polygon P of three sides, and the isoperimetric regular polygon P' of four sides.

To prove that $P' > P$.

Proof. Draw CX from C to any point X in AB.

Invert the $\triangle AXC$ and place it in the position XCY, letting X fall at C, C at X, and A at Y.

The polygon $XBCY$ is an irregular polygon of four sides, which by construction has the same perimeter as P' and the same area as P.

Then the regular polygon P' of four sides is greater than the isoperimetric irregular polygon $XBCY$ of four sides. § 418

That is, a regular polygon of four sides is greater than the isoperimetric regular polygon of three sides.

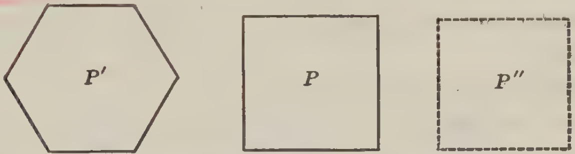
In like manner, it may be shown that P' is less than the isoperimetric regular polygon of five sides, and so on. Q.E.D.

Discussion. We may illustrate this by the case of an equilateral triangle and a square, each with the perimeter p. In the triangle the base is $\frac{1}{3}p$, the altitude $\frac{1}{6}p\sqrt{3}$, and the area $\frac{1}{36}p^2\sqrt{3}$, or about $0.048\,p^2$. In the square the base and altitude are each $\frac{1}{4}p$, and the area is $\frac{1}{16}p^2$, or $0.0625\,p^2$. The area of the polygon is therefore increasing as we increase the number of sides.

Since the limit approached by the perimeters is a circle, we may infer that *of all isoperimetric plane figures the circle has the greatest area.*

PROPOSITION IX. THEOREM

420. *Of regular polygons having a given area, that which has the greatest number of sides has the minimum perimeter.*



Given the regular polygons P and P' having the same area, P' having the greater number of sides.

To prove that the perimeter of $P >$ the perimeter of P'.

Proof. Construct the regular polygon P'' having the same perimeter as P', and the same number of sides as P.

Denote a side of P by s, and a side of P'' by s''.

Then $P' > P''$. § 419

But $P = P'$. Given

$\therefore P > P''$. Ax. 9

But $P : P'' = s^2 : s''^2$. § 374

$\therefore s^2 > s''^2$.

$\therefore s > s''$. Ax. 6

$\therefore$ the perimeter of $P >$ the perimeter of P''. Ax. 6

But the perimeter of $P' =$ the perimeter of P''. Const.

$\therefore$ the perimeter of $P >$ the perimeter of P', by Ax. 9. Q.E.D.

Discussion. We may illustrate this, as on page 270, by the case of an equilateral triangle and a square, each with area a^2. The side of the square is a, and the perimeter $4a$. The area of the equilateral triangle is $\frac{1}{4}s^2\sqrt{3}$. Therefore $\frac{1}{4}s^2\sqrt{3} = a^2$, or $\frac{1}{2}s\sqrt[4]{3} = a$. Now $\sqrt[4]{3} = \sqrt{\sqrt{3}}$; hence we have $\sqrt{3} = 1.73+$, and $\sqrt{\sqrt{3}} = \sqrt{1.73} = 1.3+$. Hence $\frac{1}{2}s \times 1.3 = a$, and $s = 1.5a$, and the perimeter of the triangle is $4.5a$. Therefore the perimeter of the square is less than that of the triangle.

EXERCISE 73

Maxima and Minima

1. Of all equivalent parallelograms that have equal bases, the rectangle has the minimum perimeter.

2. Of all equivalent rectangles, the square has the minimum perimeter.

3. Of all triangles that have the same base and the same altitude, the isosceles has the minimum perimeter.

4. Of all triangles that can be inscribed in a given circle, the equilateral is the maximum and has the maximum perimeter.

5. To inscribe in a semicircle the maximum rectangle.

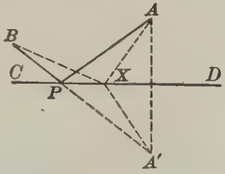
6. Find the area of the maximum triangle inscribed in a semicircle whose radius is 3 in.

7. Of all polygons of a given number of sides that can be inscribed in a given circle, that which is regular has the maximum area and the maximum perimeter.

8. Of all polygons of a given number of sides that can be circumscribed about a given circle, that which is regular has the minimum area and the minimum perimeter.

9. In a given line required to find a point such that the sum of its distances from two given points on the same side of the line shall be the minimum.

How does $AP + PB$ compare with $A'B$? and this with $A'X+XB$? and this with $AX+XB$? This is the problem of a ray of light from A to the mirror CD, and reflected to B.



10. To divide a given line into two segments such that the sum of the squares on these segments shall be the minimum.

11. To divide a given line into two segments such that their product shall be the maximum.

SOLID GEOMETRY

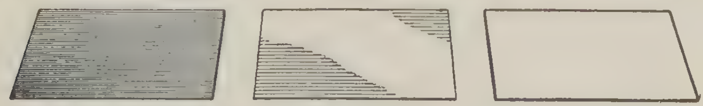
BOOK VI

LINES AND PLANES IN SPACE

421. The Nature of Solid Geometry. In plane geometry we deal with figures lying in a flat surface, studying their properties and relations and measuring the figures. In solid geometry we shall deal with figures not only of two dimensions but of three dimensions, also studying their properties and relations and measuring the figures.

422. Plane. A surface such that a straight line joining any two of its points lies wholly in the surface is called a *plane.*

A plane is understood to be indefinite in extent, but it is conveniently represented by a rectangle seen obliquely, as here shown.



423. Determining a Plane. A plane is said to be *determined* by certain lines or points if it contains the given lines or points, and no other plane can contain them.

When we suppose a plane to be drawn to include given points or lines, we are said to *pass* the plane through these points or lines.

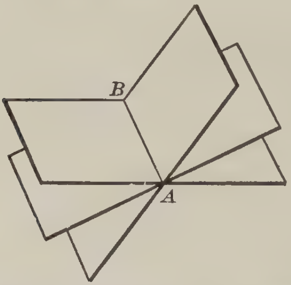
When a straight line is drawn from an external point to a plane, its point of contact with the plane is called its *foot.*

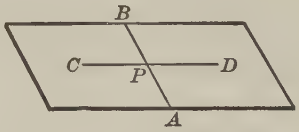
424. Intersection of Planes. The line that contains all the points common to two planes is called their *intersection.*

425. Postulate of Planes. Corresponding to the postulate that one straight line, and only one, can be drawn through two given points, the following postulate is assumed for planes:

One plane, and only one, can be passed through two given intersecting straight lines.

For it is apparent from the first figure that a plane may be made to turn about any single straight line AB, thus assuming different positions. But if CD intersects AB at P, as in the second figure, then when the plane through AB turns until it includes C, it must include D, since it includes two points, C and P, of the line (§ 422). If it turns any more, it will no longer contain C.





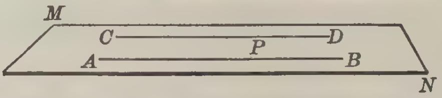
426. Corollary 1. *A straight line and a point not in the line determine a plane.*

For example, line AB and point C in the above figure.

427. Corollary 2. *Three points not in a straight line determine a plane.*

For by joining any one of them with the other two we have two intersecting lines (§ 425).

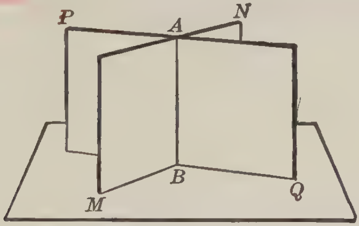
428. Corollary 3. *Two parallel lines determine a plane.*



For two parallel lines lie in a plane (§ 93), and a plane containing either parallel and a point P in the other is determined (§ 426).

PROPOSITION I. THEOREM

429. *If two planes cut each other, their intersection is a straight line.*



Given MN and PQ, two planes which cut each other.

To prove that the planes MN and PQ intersect in a straight line.

Proof. Let A and B be two points common to the two planes.

Draw a straight line through the points A and B.

Then the straight line AB lies in both planes. § 422

(*For it has two points in each plane.*)

No point not in the line AB can be in both planes; for one plane, and only one, can contain a straight line and a point without the line. § 426

Therefore the straight line through A and B contains all the points common to the two planes, and is consequently the intersection of the planes, by § 424. Q.E.D.

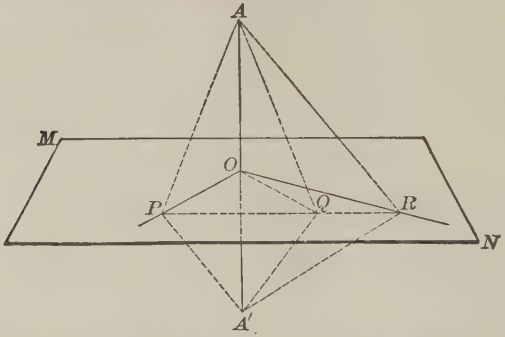
Discussion. What is the corresponding statement in plane geometry?

430. Perpendicular to a Plane. If a straight line drawn to a plane is perpendicular to every straight line that passes through its foot and lies in the plane, it is said to be *perpendicular* to the plane.

When a line is perpendicular to a plane, the plane is also said to be perpendicular to the line.

PROPOSITION II. THEOREM

431. *If a line is perpendicular to each of two other lines at their point of intersection, it is perpendicular to the plane of the two lines.*



Given the line AO perpendicular to the lines OP and OR at O.

To prove that AO is ⊥ to the plane MN of these lines.

Proof. Through O draw in MN any other line OQ, and draw PR cutting OP, OQ, OR, at P, Q, and R.

Produce AO to A', making OA' equal to OA, and join A and A' to each of the points P, Q, and R.

Then OP and OR are each ⊥ to AA' at its mid-point.

$\therefore AP = A'P$, and $AR = A'R$. § 150

$\therefore \triangle APR$ is congruent to $\triangle A'PR$. § 80

$\therefore \angle RPA = \angle A'PR$. § 67

That is, $\angle QPA = \angle A'PQ$.

$\therefore \triangle PQA$ is congruent to $\triangle PQA'$. § 68

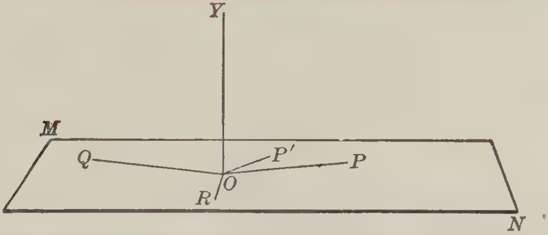
$\therefore AQ = A'Q$ (§ 67); and OQ is ⊥ to AA' at O. § 151

$\therefore AO$ is ⊥ to any and hence to every line in MN through O.

$\therefore AO$ is ⊥ to the plane MN, by § 430. Q. E. D.

PROPOSITION III. THEOREM

432. *All the perpendiculars that can be drawn to a given line at a given point lie in a plane which is perpendicular to the given line at the given point.*



Given the plane MN perpendicular to the line OY at O.

To prove that OP, any line $\perp$ to OY at O, lies in MN.

Proof. Let the plane containing OY and OP intersect the plane MN in the line OP'; then OY is $\perp$ to OP'. § 430

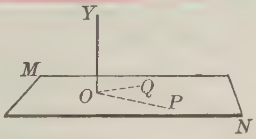
In the plane POY only one $\perp$ can be drawn to OY at O. § 57

Therefore OP and OP' coincide, and OP lies in MN.

Hence every $\perp$ to OY at O, as OQ, OR, lies in MN. Q.E.D.

433. COROLLARY 1. *Through a given point in a given line one plane, and only one, can be passed perpendicular to the line.*

434. COROLLARY 2. *Through a given external point one plane, and only one, can be passed perpendicular to a given line.*



Given the line OY and the point P.

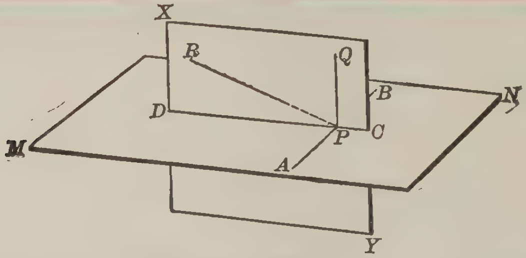
Draw $PO \perp$ to OY, and $OQ \perp$ to OY. Then OQ and OP determine a plane through $P \perp$ to OY.

Only one such plane can be drawn; for only one $\perp$ can be drawn to OY from the point P (§ 82).

435. Oblique Line. A line that meets a plane but is not perpendicular to it is said to be *oblique* to the plane.

PROPOSITION IV. THEOREM

436. *Through a given point in a plane there can be drawn one line perpendicular to the plane, and only one.*



Given the point *P* in the plane *MN*.

To prove that there can be drawn one line perpendicular to the plane MN at P, and only one.

Proof. Through the point P draw in the plane MN any line AB, and pass through P a plane $XY \perp$ to AB, cutting the plane MN in CD. § 433

At P erect in the plane XY the line $PQ \perp$ to CD.

The line AB, being $\perp$ to the plane XY by construction, is $\perp$ to PQ, which passes through its foot in the plane. § 430

That is, PQ is $\perp$ to AB; and as it is $\perp$ to CD by construction, it is $\perp$ to the plane MN. § 431

Moreover, any other line PR drawn from P is oblique to MN. For PQ and PR intersecting in P determine a plane.

To avoid drawing another plane, use XY again to represent the plane of PQ and PR, letting it cut MN in the line CD.

Then since PQ is $\perp$ to MN, it is $\perp$ to CD. § 430

Therefore PR is oblique to CD. § 57

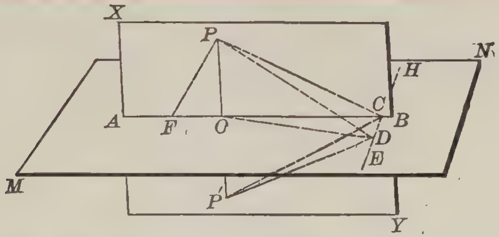
Therefore PR is oblique to MN. § 435

Therefore PQ is the only $\perp$ to MN at the point P. Q.E.D.

Discussion. What is the corresponding proposition in plane geometry?

PROPOSITION V. THEOREM

437. *Through a given external point there can be drawn one line perpendicular to a given plane, and only one.*



Given the plane *MN* and the external point *P*.

To prove that there can be drawn one line from P perpendicular to the plane MN, and only one.

Proof. In MN draw any line EH, and let XY be a plane through $P \perp$ to EH, cutting MN in AB, and EH in C.

Draw $PO \perp$ to AB, and in MN draw any line OD from O to EH.

Produce PO, making $OP' = OP$, and draw PC, PD, $P'C$, $P'D$.

Since DC is $\perp$ to XY, $\angle$s PCD and $P'CD$ are right angles. § 430

Since the side DC is common, and $PC = P'C$, § 150

$\therefore$ rt. $\triangle PCD$ is congruent to rt. $\triangle P'CD$. § 69

$\therefore PD = P'D$. § 67

$\therefore OD$ is $\perp$ to PP' at O. § 151

$\therefore PO$ is $\perp$ to MN, being $\perp$ to OD and AB. § 431

Moreover, every other line PF from P to MN is oblique to MN. (The proof is left for the student.)

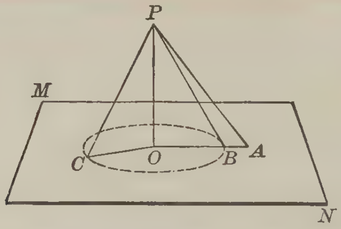
$\therefore PO$ is the only $\perp$ from P to MN. Q.E.D.

438. COROLLARY. *The perpendicular is the shortest line from a point to a plane.*

The length of this $\perp$ is called the *distance* from the point to the plane.

PROPOSITION VI. THEOREM

439. *Oblique lines drawn from a point to a plane, meeting the plane at equal distances from the foot of the perpendicular, are equal; and of two oblique lines, meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.*



Given the plane *MN*, the perpendicular line *PO*, the oblique lines *PA*, *PB*, *PC*, the equal distances *OB*, *OC*, and the unequal distances *OA*, *OC*, with *OA* greater than *OC*.

To prove that $PB = PC$, *and* $PA > PC$.

Proof. In the $\triangle s$ OBP and OCP,

$OP = OP$, Iden.

$OB = OC$, Given

and $\angle BOP = \angle POC$. § 56

$\therefore \triangle OBP$ is congruent to $\triangle OCP$. § 69

$\therefore PB = PC$. § 67

Let A, B, and O lie in the same straight line.

Then $OA > OC$. Given

$\therefore OA > OB$. Ax. 9

$\therefore PA > PB$. § 84

$\therefore PA > PC$, by Ax. 9. Q.E.D.

Discussion. Compare the corresponding case in plane geometry.

440. Corollary 1. *Equal oblique lines drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

In the figure on page 280, if PB is given equal to PC, then since $PO = PO$, and the angles at O are right angles, what follows with respect to the △ OBP and OCP? with respect to OB and OC?

Furthermore, if $PA > PC$, how does PA compare with PB?

Then how does OA compare with OB? Why?

Then how does OA compare with OC?

441. Corollary 2. *The locus of a point equidistant from all points on a circle is a line through the center, perpendicular to the plane of the circle.*

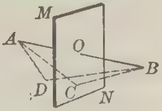
In the figure on page 280, in order to prove that PO is the required locus what must be proved for any point on PO (§ 148)? for any point not on PO? Prove both of these facts.

442. Corollary 3. *The locus of a point equidistant from the vertices of a triangle is a line through the center of the circumscribed circle, perpendicular to the plane of the triangle.*

How does this follow from Corollary 2?

What locus is the line through the center of the inscribed circle, perpendicular to the plane of the triangle?

443. Corollary 4. *The locus of a point equidistant from two given points is the plane perpendicular to the line joining them, at its mid-point.*



For any point C in this plane lies in a ⊥ to AB at O, its mid-point (§ 430).

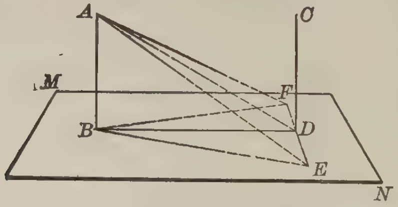
Hence how do CA and CB compare (§ 150)?

And any point D outside the plane MN cannot lie in a ⊥ to AB at O. What may therefore be said as to the distances from D to A and B (§ 150)?

What is the proposition in plane geometry corresponding to Corollary 4? In what respect do the two proofs differ?

PROPOSITION VII. THEOREM

444. *Two lines perpendicular to the same plane are parallel.*



Given the lines *AB* and *CD*, perpendicular to the plane *MN*.

To prove that *AB and CD are parallel.*

Proof. Draw *AD* and *BD*, and in *MN* draw through *D* *EF* ⊥ to *BD*, making *DE* = *DF*. Draw *BE*, *AE*, *BF*, *AF*.

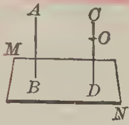
Now prove that △ *BDE* and *BDF* are congruent (§ 69), that ∡ *ADE* and *ADF* are right angles (§ 80), and that *BD*, *CD*, and *AD* lie in the same plane (§ 432).

But *AB* also lies in this plane, § 422

and *AB* and *CD* are both ⊥ to *BD*. § 430

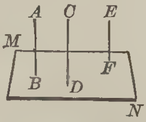
∴ *AB* is ∥ to *CD*, by § 95. Q.E.D.

445. COROLLARY 1. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



For if through any point *O* of *CD* a line is drawn ⊥ to *MN*, how is it related to *AB* (§ 444)? Now apply § 94.

446. COROLLARY 2. *If two lines are parallel to a third line, they are parallel to each other.*



For a plane *MN* ⊥ to *CD* is ⊥ to *AB* and *EF* (§ 445).

447. Line and Plane Parallel. If a line and plane cannot meet, however far produced, they are said to be *parallel*.

EXERCISE 74

1. Why does folding a sheet of paper give a straight edge?

2. If equal oblique lines are drawn from a given external point to a plane, they make equal angles with lines drawn from the points where the oblique lines meet the plane to the foot of the perpendicular from the given point.

3. If from the foot of a perpendicular to a plane a line is drawn at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to any point of the perpendicular is perpendicular to the line of the plane.

4. If two perpendiculars are drawn from a point to a plane and to a line in that plane respectively, the line joining the feet of the perpendiculars is perpendicular to the given line.

5. From two vertices of a triangle perpendiculars are let fall on the opposite sides. From the intersection of these perpendiculars a perpendicular is drawn to the plane of the triangle. Prove that a line drawn to any vertex of the triangle, from any point on this perpendicular, is perpendicular to the line drawn through that vertex parallel to the opposite side.

6. Find the point in a plane to which lines may be drawn from two given external points on the same side of the plane so that their sum shall be the least possible.

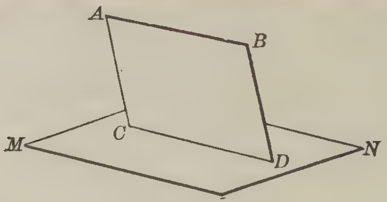
From one point A suppose a $\perp$ AO drawn to the plane and produced to A', making $OA' = OA$. Connect A' and the other point B by a line cutting the plane at P. Then BPA is the shortest line.

7. If three equal oblique lines are drawn from an external point to a plane, the perpendicular from the point to the plane meets the plane at the center of the circle circumscribed about the triangle having for its vertices the feet of the oblique lines.

8. State and prove the propositions of plane geometry corresponding to §§ 444, 445, and 446. Why do not the proofs of those propositions apply to these sections?

Proposition VIII. Theorem

448. *If two lines are parallel, every plane containing one of the lines, and only one, is parallel to the other line.*



Given the parallel lines *AB* and *CD*, and the plane *MN* containing *CD* but not *AB*.

To prove that the plane MN is parallel to AB.

Proof. *AB* and *CD* are in the same plane, *AD*. § 93

This plane *AD* intersects the plane *MN* in *CD*. Given

Now *AB* lies in the plane *AD*, however far produced. § 422

Therefore, if *AB* meets the plane *MN* at all, the point of meeting must be in the line *CD*. § 422

But since *AB* is ∥ to *CD*, Given

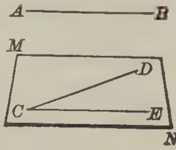
∴ *AB* cannot meet *CD*. § 93

∴ *AB* cannot meet the plane *MN*.

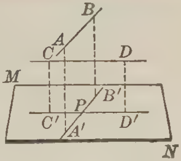
∴ *MN* is ∥ to *AB*, by § 447. Q.E.D.

449. Corollary 1. *Through either of two lines not in the same plane one plane, and only one, can be passed parallel to the other.*

For if *AB* and *CD* are the lines, and we pass a plane through *CD* and a line *CE* which is drawn parallel to *AB*, what can be said of the plane *MN* determined by *CD* and *CE*, with respect to the line *AB*? Why can there be only one such plane?



450. Corollary 2. *Through a given point one plane, and only one, can be passed parallel to any two given lines in space.*



Suppose P the given point and AB and CD the given lines. If, now, we draw through P the line $A'B'$ parallel to AB, and the line $C'D'$ parallel to CD, these lines will determine the plane MN (§ 425). Then what may be said of the plane MN with respect to the lines AB and CD? Why can only one plane be so passed through P?

Discussion. Proposition VIII might of course be made more general by allowing both of the parallels to lie in the plane MN. That is, *If two lines are parallel, a plane containing one of the lines cannot intersect the other*, although the other line might lie in it.

In the figure of Corollary 2 the $\angle D'PB'$ is sometimes spoken of as the angle between the nonintersecting lines AB and CD, although this is not commonly done in elementary geometry.

451. Parallel Planes. Two planes which cannot meet, however far produced, are said to be *parallel.*

EXERCISE 75

1. What is the locus of a point in a plane equidistant from two parallel lines? What is the corresponding locus in space, given two parallel planes instead of two parallel lines? Draw the figure, without proof.

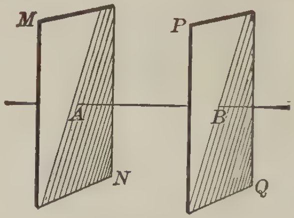
2. Find the locus in a plane of a point at a given distance from a given external point. What is the corresponding case of plane geometry?

3. If a given line is parallel to a given plane, the intersection of the plane with any plane passed through the given line is parallel to that line.

4. If a given line is parallel to a given plane, a line parallel to the given line drawn through any point of the plane lies in the plane.

PROPOSITION IX. THEOREM

452. *Two planes perpendicular to the same line are parallel.*



Given the planes *MN* and *PQ* perpendicular to the line *AB*.

To prove that the planes MN and PQ are parallel.

Proof. If MN and PQ are not parallel, they must meet.

If they could meet, we should have two planes from a point of their intersection ⊥ to the same straight line.

But this is impossible. § 434

∴ MN and PQ are parallel, by § 451. Q.E.D.

EXERCISE 76

1. What is the locus of a point equidistant from two given points A, B, and also equidistant from two other given points C, D?

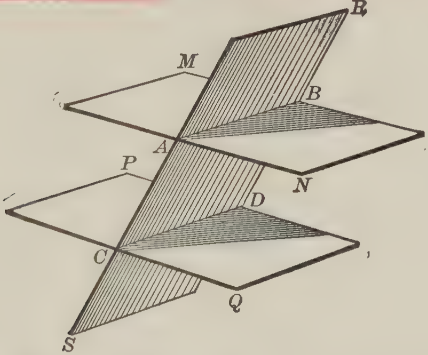
2. What is the locus of a point at the distance d from a given plane P, and at the distance d' from a given plane P'?

3. What is the locus of a point at the distance d from a given plane P, and equidistant from two given points A, B?

4. Find a point at the distance d from a given plane P, at the distance d' from a given plane P', and equidistant from two given points A, B. Can there be more than one such point? Draw the figure, without proof.

Proposition X. Theorem

453. *The intersections of two parallel planes by a third plane are parallel lines.*



Given the parallel planes *MN* and *PQ*, cut by the plane *RS* in *AB* and *CD* respectively.

To prove that the intersections AB and CD are parallel.

Proof. AB and CD are in the same plane RS. Given

If AB and CD meet, the planes MN and PQ must meet, since AB is always in MN and CD is always in PQ. § 422

But MN and PQ cannot meet. § 451

$\therefore AB$ is ∥ to CD, by § 93. Q.E.D.

454. Corollary 1. *Parallel lines included between parallel planes are equal.*

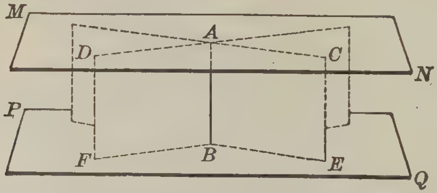
In the above figure, suppose AC ∥ to BD. Then the plane of AC and BD will intersect MN and PQ in lines that are how related to each other? Then what kind of a figure is $ACDB$?

455. Corollary 2. *Two parallel planes are everywhere equidistant from each other.*

Drop perpendiculars from any points in MN to PQ. Prove that these perpendiculars are parallel and hence (§ 454) that they are equal.

PROPOSITION XI. THEOREM

456. *A line perpendicular to one of two parallel planes is perpendicular to the other also.*



Given the line AB perpendicular to the plane MN, and the plane PQ parallel to the plane MN.

To prove that AB is perpendicular to the plane PQ.

Proof. Pass through AB two planes AE, AF, intersecting MN in AC, AD, and intersecting PQ in BE, BF, respectively.

Then AC is ∥ to BE, and AD is ∥ to BF. § 453

But AB is ⊥ to AC and AD. § 430

∴ AB is ⊥ to BE and BF. § 97

∴ AB is ⊥ to the plane PQ, by § 431. Q.E.D.

457. COROLLARY 1. *Through a given point one plane, and only one, can be passed parallel to a given plane.*

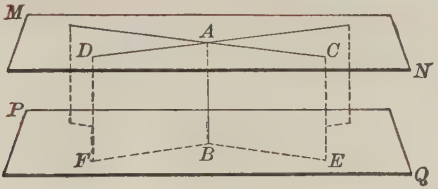
How is a plane through A, ⊥ to AB, related to PQ? Now use § 433.

458. COROLLARY 2. *The locus of a point equidistant from two parallel planes is a plane perpendicular to a line which is perpendicular to the planes and which bisects the segment cut off by them.*

459. COROLLARY 3. *The locus of a point equidistant from two parallel lines is a plane perpendicular to a line which is perpendicular to the given lines and which bisects the segment cut off by them.*

PROPOSITION XII. THEOREM

460. *If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to that plane.*



Given the intersecting lines *AC*, *AD*, each parallel to the plane *PQ*, and let *MN* be the plane determined by *AC* and *AD*.

To prove that *MN is parallel to PQ.*

Proof. Draw $AB \perp$ to PQ.

Pass a plane through AB and AC intersecting PQ in BE, and a plane through AB and AD intersecting PQ in BF.

Then AB is $\perp$ to BE and BF. § 430

But AC and BE lie in the same plane, Const.

and AC cannot meet BE without meeting the plane PQ, which is impossible. § 447

$\therefore BE$ is $\parallel$ to AC. § 93

Similarly BF is $\parallel$ to AD.

$\therefore AB$ is $\perp$ to AC and to AD. § 97

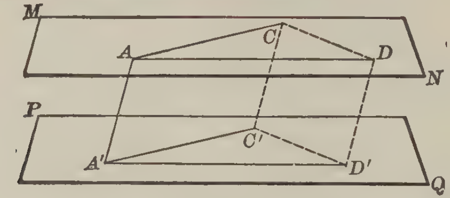
$\therefore AB$ is $\perp$ to the plane MN. § 431

$\therefore MN$ is $\parallel$ to PQ, by § 452. Q.E.D.

Discussion. It is evident that this proposition does not depend upon the position of A. For example, C and D might remain where they are and A might recede a long distance, AC and AD becoming more nearly parallel. So long as the lines intersect, and only so long, are we certain that the planes are parallel.

PROPOSITION XIII. THEOREM

461. *If two angles not in the same plane have their sides respectively parallel and lying on the same side of the straight line joining their vertices, the angles are equal, and their planes are parallel.*



Given the angles A and A', in the planes MN and PQ respectively, and their corresponding sides parallel and lying on the same side of AA'.

To prove that $\angle A = \angle A'$, and that MN is ∥ to PQ.

Proof. Take AD and $A'D'$ equal, also AC and $A'C'$ equal.

Draw DD', CC', CD, $C'D'$.

Since AD is equal and ∥ to $A'D'$,

$\therefore AA'$ is equal and ∥ to DD'. § 130

In like manner AA' is equal and ∥ to CC'.

$\therefore DD'$ and CC' are equal, Ax. 8

and DD' and CC' are parallel. § 446

$\therefore CD = C'D'$. § 130

$\therefore \triangle ADC$ is congruent to $\triangle A'D'C'$. § 80

$\therefore \angle A = \angle A'$. § 67

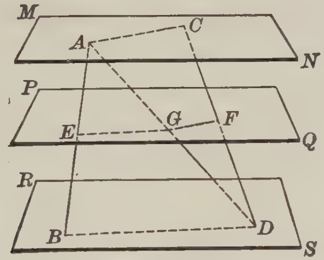
But MN is ∥ to each of the lines $A'C'$ and $A'D'$. § 448

$\therefore MN$ is ∥ to PQ, by § 460. Q.E.D.

Discussion. Why does not the proof of the corresponding proposition in plane geometry apply here?

PROPOSITION XIV. THEOREM

462. *If two lines are cut by three parallel planes, their corresponding segments are proportional.*



Given the lines AB and CD, cut by the parallel planes MN, PQ, RS, in the points A, E, B, and C, F, D, respectively.

To prove that $AE : EB = CF : FD.$

Proof. Draw AD cutting the plane PQ in G.

Pass a plane through AB and AD, intersecting PQ in the line EG, and intersecting RS in the line BD.

Also pass a plane through AD and CD, intersecting PQ in the line GF, and intersecting MN in the line AC.

Then EG is ∥ to BD,

and GF is ∥ to AC. § 453

$\therefore AE : EB = AG : GD,$

and $CF : FD = AG : GD.$ § 273

$\therefore AE : EB = CF : FD$, by Ax. 8. Q.E.D.

Discussion. This is a generalization of § 275. It may be stated still more generally, *If two lines are cut by any number of parallel planes, their corresponding segments are proportional.* In particular, the case might be considered in which AB and CD intersect between the planes.

Why does not the proof of the corresponding case (§ 275) in plane geometry apply here?

EXERCISE 77

1. Find the locus of a line drawn through a given point, parallel to a given plane.

2. Find the locus of a point in a given plane that is equidistant from two given points not in the plane.

3. Find the locus of a point equidistant from three given points not in a straight line.

4. Find the locus of a point equidistant from two given parallel planes and also equidistant from two given points.

5. What is the locus of a point in a plane at a given distance from a given line in the plane? What is the locus of a point at a given distance from a given plane?

6. The line AB cuts three parallel planes in the points A, E, B; and the line CD cuts these planes in the points C, F, D. If $AE = 6$ in., $EB = 8$ in., and $CD = 12$ in., compute CF and FD.

7. The line AB cuts three parallel planes in the points A, E, B; and the line CD cuts these planes in the points C, F, D. If $AB = 8$ in., $CF = 5$ in., and $CD = 9$ in., compute AE and EB.

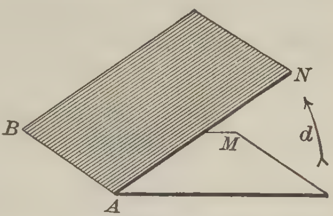
8. To draw a perpendicular to a given plane from a given point without the plane.

9. To erect a perpendicular to a given plane at a given point in the plane.

10. It is proved in plane geometry that if three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal. State and prove a corresponding proposition in solid geometry.

11. It is proved in plane geometry that the line joining the mid-points of two sides of a triangle is parallel to the third side. State and prove a corresponding proposition in solid geometry, referring to a plane passing through the mid-points of two sides of a triangle.

463. Dihedral Angle. The opening between two intersecting planes is called a *dihedral angle.*

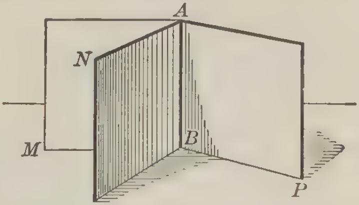


In this figure the two planes *AM* and *BN* are called the *faces* of the dihedral angle, and the line of intersection *AB* is called the *edge* of the angle.

A dihedral angle is read by naming the letters designating its edge, or its faces and edge, or by a small letter within. Thus the dihedral angle here shown may be designated by *AB*, *M-AB-N*, or *d*.

464. Size of a Dihedral Angle. The size of a dihedral angle depends upon the amount of turning necessary to bring one face into the position of the other.

The analogy to the plane angle is apparent, and is still further seen as we proceed.



465. Adjacent Dihedral Angles. If two dihedral angles have a common edge, and a common face between them, they are said to be *adjacent dihedral angles.*

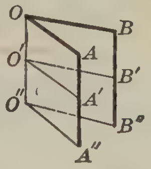
For example, *M-AB-N* and *N-BA-P* are adjacent dihedral angles.

466. Right Dihedral Angle. If one plane meets another plane and makes the adjacent dihedral angles equal, each of these angles is called a *right dihedral angle.*

Dihedral angles are said to be *straight*, *acute*, *obtuse*, *reflex*, *complementary*, *supplementary*, *conjugate*, and *vertical*, under conditions similar to those obtaining with plane angles. There is little occasion, however, to use any of these terms in connection with dihedral angles.

467. Perpendicular Planes. If two planes intersect and form a right dihedral angle, each of the planes is said to be *perpendicular* to the other plane.

468. Plane Angle of a Dihedral Angle. The plane angle formed by two straight lines, one in each plane, perpendicular to the edge at the same point, is called the *plane angle of the dihedral angle.*



For example, $\angle AOB$ is the plane angle of the dihedral angle OO'', if AO and BO are each $\perp$ to OO''.

469. Corollary. *The plane angle of a dihedral angle has the same magnitude from whatever point in the edge the perpendiculars are drawn.*

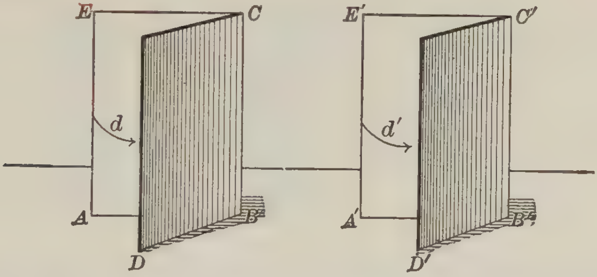
How is $O'B'$ related to OB, and $O'A'$ to OA (§ 95)? Then how is $\angle A'O'B'$ related to $\angle AOB$ (§ 461)?

470. Relation of Dihedral Angles to Plane Angles. It is apparent that the demonstrations of many properties of dihedral angles are identically the same as the demonstrations of analogous properties of plane angles. A few of the more important propositions will be proved, but the following may be assumed or may be taken as exercises:

1. If a plane meets another plane, it forms with it two adjacent dihedral angles whose sum is equal to two right dihedral angles.
2. If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are in the same plane.
3. If two planes intersect each other, their vertical dihedral angles are equal.
4. If a plane intersects two parallel planes, the alternate-interior dihedral angles are equal; the exterior-interior dihedral angles are equal; and the two interior dihedral angles on the same side of the transverse plane are supplementary.
5. When two planes are cut by a third plane, if the alternate-interior dihedral angles are equal, or the exterior-interior dihedral angles are equal, and the edges of the dihedral angles thus formed are parallel, the two planes are parallel.
6. Two dihedral angles whose faces are parallel each to each are either equal or supplementary.
7. Two dihedral angles whose faces are perpendicular each to each, and whose edges are parallel, are either equal or supplementary.

Proposition XV. Theorem

471. *Two dihedral angles are equal if their plane angles are equal.*



Given two equal plane angles ABD and $A'B'D'$ of the two dihedral angles d and d'.

To prove that the dihedral angles d and d' are equal.

Proof. Apply dihedral angle d' to dihedral angle d, making the plane $\angle A'B'D'$ coincide with its equal $\angle ABD$.

Then since $B'C'$ is $\perp$ to $A'B'$ and $D'B'$, § 468

$\therefore B'C'$ is $\perp$ to the plane $A'B'D'$. § 431

$\therefore B'C'$ will also be $\perp$ to the plane ABD at B. Post. 5

$\therefore B'C'$ will fall on BC. § 436

Then the planes $A'B'C'$ and ABC, having in common the two intersecting lines AB and BC, coincide. § 425

In the same way it may be shown that the planes $D'B'C'$ and DBC coincide.

Therefore the two dihedral angles d and d' coincide and are equal. Q.E.D.

Discussion. May we have equal straight dihedral angles? equal reflex dihedral angles? What is the authority for saying that right dihedral angles are equal?

Proposition XVI. Theorem

472. *Two dihedral angles have the same ratio as their plane angles.*

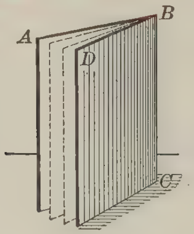


Fig. 1

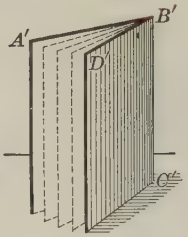


Fig. 2

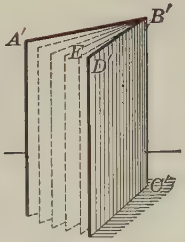


Fig. 3

Given two dihedral angles BC and $B'C'$, and let their plane angles be ABD and $A'B'D'$ respectively.

To prove that $\angle B'C' : \angle BC = \angle A'B'D' : \angle ABD$.

Case 1. *When the plane angles are commensurable.*

Proof. Suppose the $\angle$s ABD and $A'B'D'$ (Figs. 1 and 2) have a common measure, which is contained m times in $\angle ABD$ and n times in $\angle A'B'D'$.

Then $\angle A'B'D' : \angle ABD = n : m$.

Apply this measure to $\angle ABD$ and $\angle A'B'D'$, and through the lines of division and the edges BC and $B'C'$ pass planes.

These planes divide $\angle BC$ into m parts, and $\angle B'C'$ into n parts, equal each to each. § 471

$$\therefore \angle B'C' : \angle BC = n : m.$$

$\therefore \angle B'C' : \angle BC = \angle A'B'D' : \angle ABD$, by Ax. 8. Q.E.D.

As with plane angles, there is also the case of incommensurables. Since the common measure may be taken as small as we please, it is evident that for practical purposes the above proof is sufficient. The proof for the incommensurable case, p. 297, may be omitted at the discretion of the teacher without destroying the sequence.

CASE 2. *When the plane angles are incommensurable.*

Proof. Divide the $\angle ABD$ into any number of equal parts, and apply one of these parts to the $\angle A'B'D'$ (Figs. 1 and 3) as a unit of measure.

Since $\angle ABD$ and $\angle A'B'D'$ are incommensurable, a certain number of these parts will form the $\angle A'B'E$, leaving a remainder $\angle EB'D'$, less than one of the parts.

Pass a plane through $B'E$ and $B'C'$.

Since the plane angles of the dihedral angles $A\text{-}BC\text{-}D$ and $A'\text{-}B'C'\text{-}E$ are commensurable,

$$\therefore A'\text{-}B'C'\text{-}E : A\text{-}BC\text{-}D = \angle A'B'E : \angle ABD. \qquad \text{Case 1}$$

By increasing the number of equal parts into which $\angle ABD$ is divided we can diminish the magnitude of each part, and therefore can make the $\angle EB'D'$ less than any assigned positive value, however small.

Hence the $\angle EB'D'$ approaches zero as a limit, as the number of parts is indefinitely increased, and at the same time the corresponding dihedral $\angle E\text{-}B'C'\text{-}D'$ approaches zero as a limit. § 204

Therefore the $\angle A'B'E$ approaches the $\angle A'B'D'$ as a limit, and the $\angle A'\text{-}B'C'\text{-}E$ approaches the $\angle A'\text{-}B'C'\text{-}D'$ as a limit.

$$\therefore \text{the variable } \frac{\angle A'B'E}{\angle ABD} \text{ approaches } \frac{\angle A'B'D'}{\angle ABD} \text{ as a limit,}$$

$$\text{and the variable } \frac{\angle A'\text{-}B'C'\text{-}E}{\angle A\text{-}BC\text{-}D} \text{ approaches } \frac{\angle A'\text{-}B'C'\text{-}D'}{\angle A\text{-}BC\text{-}D} \text{ as a limit.}$$

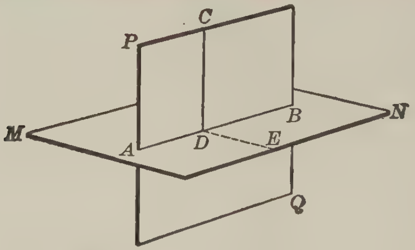
But $\dfrac{\angle A'B'E}{\angle ABD}$ is always equal to $\dfrac{\angle A'\text{-}B'C'\text{-}E}{\angle A\text{-}BC\text{-}D}$, as $\angle A'B'E$ varies in value and approaches $\angle A'B'D'$ as a limit. Case 1

$$\therefore \frac{\angle A'\text{-}B'C'\text{-}D'}{\angle A\text{-}BC\text{-}D} = \frac{\angle A'B'D'}{\angle ABD}, \text{ by § 207.} \qquad \text{Q.E.D}$$

473. COROLLARY. *The plane angle of a dihedral angle may be taken as the measure of the dihedral angle.*

PROPOSITION XVII. THEOREM

474. *If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Given the planes *MN* and *PQ* perpendicular to each other, and the line *CD* in *PQ* perpendicular to their intersection *AB*.

To prove that CD is perpendicular to the plane MN.

Proof. In the plane *MN* draw *DE* ⊥ to *AB* at *D*.

Then $\angle EDC$ is a right angle, § 473

and $\angle CDA$ is also a right angle. Given

∴ *CD* is ⊥ to the plane *MN*, by § 431. Q.E.D.

475. COROLLARY 1. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection will lie in the other.*

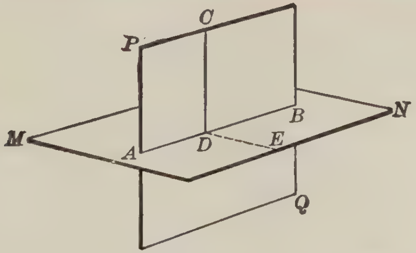
Will a line *CD* drawn in the plane *PQ* ⊥ to *AB* at *D* be ⊥ to the plane *MN*? How many ⊥s can be drawn from *D* to the plane *MN*?

476. COROLLARY 2. *If two planes are perpendicular to each other, a perpendicular to the first from any point in the second will lie in the second.*

Will a line *CD* drawn in the plane *PQ* from *C* ⊥ to *AB* be ⊥ to the plane *MN*? How many ⊥s can be drawn from *C* to the plane *MN*?

PROPOSITION XVIII. THEOREM

477. *If a line is perpendicular to a plane, every plane passed through this line is perpendicular to the plane.*



Given the line *CD* perpendicular to the plane *MN* at the point *D*, and *PQ* any plane passed through *CD* intersecting *MN* in *AB*.

To prove that the plane PQ is perpendicular to the plane MN.

Proof. Draw DE in the plane $MN \perp$ to AB.

Since	CD is $\perp$ to MN,	Given
	$\therefore CD$ is $\perp$ to AB.	§ 430
	$\therefore \angle EDC$ measures $\angle N\text{-}AB\text{-}P$.	§ 473
	But $\angle EDC$ is a right angle.	§ 430
	$\therefore PQ$ is $\perp$ to MN, by § 467.	Q.E.D.

EXERCISE 78

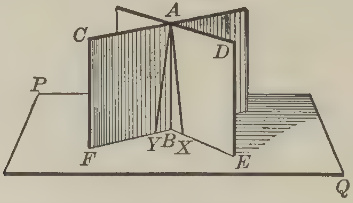
1. A plane perpendicular to the edge of a dihedral angle is perpendicular to each of its faces.

2. If one line is perpendicular to another, is any plane passed through the first line perpendicular to the second? Prove it.

3. If three lines are perpendicular to one another at a common point, what is the relation to one another of the three planes determined by the three pairs of lines? Prove it.

PROPOSITION XIX. THEOREM

478. *If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.*



Given two planes *BC* and *BD*, intersecting in *AB*, and each perpendicular to the plane *PQ*.

To prove that AB is perpendicular to the plane PQ.

Proof. Let the plane *BC* intersect the plane *PQ* in *BF*,

and let the plane *BD* intersect the plane *PQ* in *BE*.

From any point *A* on *AB* draw *AX* ⊥ to *BE*,

and from *A* draw *AY* ⊥ to *BF*.

Then *AX* and *AY* are both ⊥ to the plane *PQ*. § 474

But it is impossible to draw two ⊥s to the plane *PQ*

from a point outside the plane *PQ*, § 437

or from a point in the plane *PQ*. § 436

∴ *AX* and *AY* must coincide.

But *AX* and *AY* can coincide only if they lie in both planes.

And all points common to both planes lie in *AB*. § 429

∴ *AX* and *AY* coincide with *AB*.

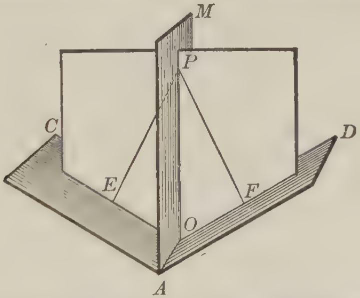
∴ *AB* is ⊥ to the plane *PQ*. Q.E.D.

Discussion. How does it appear from this proof that *AB* cannot be parallel to *PQ*?

The proposition is illustrated in the intersection of two walls of a room with the floor or the ceiling.

PROPOSITION XX. THEOREM

479. *The locus of a point equidistant from the faces of a dihedral angle is the plane bisecting the angle.*



Given the plane *AM* bisecting the dihedral angle formed by the planes *AD* and *AC*.

To prove that the plane AM is the locus of a point equidistant from the planes AD and AC.

Proof. Let EOF be a plane $\perp$ to AO, the intersection of the planes AD and AC, at O.

Since AO is $\perp$ to the plane EOF,

$\therefore$ the planes AD, AM, and AC are $\perp$ to the plane EOF. § 477

From any point P, in the intersection of the planes AM and EOF, draw $PF \perp$ to OF, and $PE \perp$ to OE.

Then PF is $\perp$ to AD, and PE is $\perp$ to AC. § 474

$\therefore$ PF and PE measure the distances from the point P to the planes AD and AC. § 438

Since AO is $\perp$ to OF, OP, and OE, § 430

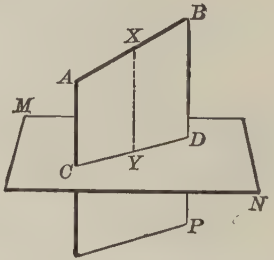
$\therefore$ OP bisects $\angle FOE$. § 473

$\therefore$ OP is the locus of a point equidistant from OF and OE. § 152

$\therefore$ AM, which contains all points P, is the locus of a point equidistant from the planes AD and AC. Q.E.D.

PROPOSITION XXI. THEOREM

480. *Through a given line not perpendicular to a given plane, one plane and only one can be passed perpendicular to the plane.*



Given the line *AB* not perpendicular to the plane *MN*.

To prove that one plane can be passed through AB perpendicular to the plane MN, and only one.

Proof. From any point X of AB draw $XY \perp$ to the plane MN, and through AB and XY pass a plane AP. § 425

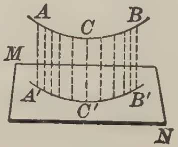
The plane AP is $\perp$ to the plane MN, since it passes through XY, a line $\perp$ to MN. § 477

Moreover, if two planes could be passed through $AB \perp$ to the plane MN, their intersection AB would be $\perp$ to MN. § 478

But this is impossible, since AB is not $\perp$ to MN. Given

Hence one plane can be passed through $AB \perp$ to the plane MN, and only one. Q.E.D.

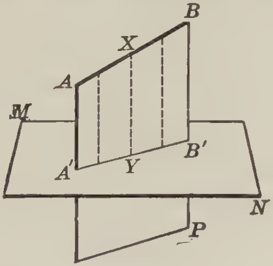
481. Projection of a Point. The foot of the line from a given point perpendicular to a plane is called the *projection of the point* on the plane.



482. Projection of a Line. The locus of the projections of the points of a line on a plane is called the *projection of the line* on the plane.

PROPOSITION XXII. THEOREM

483. *The projection of a straight line not perpendicular to a plane, upon that plane, is a straight line.*



Given the straight line *AB* not perpendicular to the plane *MN*, and *A'B'* the projection of *AB* upon *MN*.

To prove that *A'B' is a straight line.*

Proof. From any point X of AB draw $XY \perp$ to MN,

and pass a plane AP through XY and AB. § 425

The plane AP is $\perp$ to the plane MN, § 477

and contains all the ⊥s drawn from AB to MN. § 476

Hence $A'B'$ must be the intersection of these two planes.

Therefore $A'B'$ is a straight line, by § 429. Q.E.D.

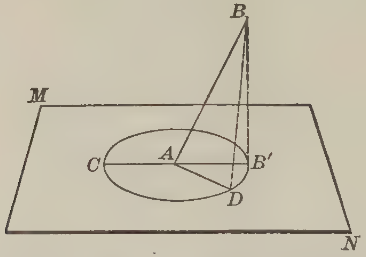
484. COROLLARY. *The projection of a straight line perpendicular to a plane, upon that plane, is a point.*

485. Inclination of a Line. The angle which a line makes with its projection on a plane is considered as the angle which it makes with the plane, and is called the *inclination of the line* to the plane.

Therefore a line ordinarily makes an acute angle with a plane, since it makes an acute angle with its projection on the plane. The cases of perpendicular and parallel lines have already been considered.

PROPOSITION XXIII. THEOREM

486. *The acute angle which a line makes with its projection upon a plane is the least angle which it makes with any line of the plane.*



Given the line AB meeting the plane MN at A, AB' being the projection of AB upon the plane MN, and AD being any other line drawn through A in the plane MN.

To prove that $\angle B'AB$ is less than $\angle DAB$.

Proof. Make AD equal to AB', and draw BB' and BD.

Then in $\triangle$s BAB' and BAD,

$AB = AB$, Iden.

$AB' = AD$, Const.

and $BB' < BD$. § 438

$\therefore \angle B'AB < \angle DAB$, by § 116. Q.E.D.

Discussion. Since $\angle B'AB$ is the least angle that AB makes with any line of the plane, how does $\angle BAC$ compare with the angles that AB makes with other lines of the plane? State the general proposition involved in the answer.

If AB is parallel to the plane, what interpretation may be given to the proposition?

If AB is perpendicular to the plane, what interpretation may be given to the proposition?

As AD swings around from the position AB' to the position AC, what kind of change takes place in the angle DAB?

EXERCISE 79

1. Describe the position of a segment of a line relative to a given plane if the projection of the segment on the plane is equal to its own length.

2. From a point A, 4 in. from a plane MN, an oblique line AC 5 in. long is drawn to the plane and made to turn around the perpendicular AB dropped from A to the plane. Find the area of the circle described by the point C.

3. From a point A, 8 in. from a plane MN, a perpendicular AB is drawn to the plane; with B as a center and a radius equal to 6 in., a circle is described in the plane; at any point C on this circle a tangent CD is drawn 24 in. in length. Find the distance from A to D.

4. Equal lines drawn from a given external point to a given plane are equally inclined to the plane.

5. If three equal lines are drawn to a plane from an external point, the perpendicular from the point to the plane determines the center of the circle circumscribed about the triangle determined by the planes of the three lines.

6. Three lines not in the same plane meet in a point. How shall a line be drawn so as to make equal angles with all three of these lines?

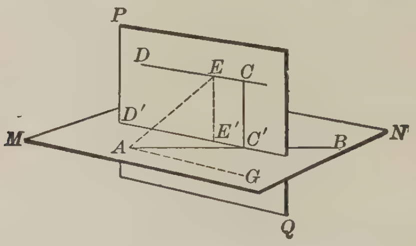
7. From a point P two perpendiculars PX and PY are drawn to two planes MN and AC which intersect in AB. From Y a perpendicular YZ is drawn to MN. Prove that the line XZ is perpendicular to AB.

8. If the length of the shadow of a tree standing on level ground exceeds the height of the tree, the angle made by the sun above the horizon must be less than what known angle?

9. Find the locus of a point at a given distance from a given plane and equidistant from two given points not in the plane.

PROPOSITION XXIV. THEOREM

487. *Between two lines not in the same plane there can be one common perpendicular, and only one.*



Given AB and CD, two lines not in the same plane.

To prove that there can be one common perpendicular, and only one, between AB and CD.

Proof. Through any point A of AB draw $AG \parallel$ to DC.

Let MN be the plane determined by AB and AG. § 425

Then the plane MN is $\parallel$ to DC. § 448

Through DC pass the plane $PQ \perp$ to the plane MN. § 480

Then DC cannot meet $D'C'$, since it is $\parallel$ to the plane MN and lies in the plane PQ. § 422

$\therefore$ DC is $\parallel$ to $D'C'$. § 93

$\therefore$ if AB is $\parallel$ to $D'C'$ it must be $\parallel$ to DC. § 446

But AB is not $\parallel$ to DC, for they are not in the same plane. Given

$\therefore$ AB must intersect $D'C'$ at some point as C'.

Draw $C'C \perp$ to the plane MN.

Then $C'C$ is $\perp$ to AB and to $D'C'$. § 430

Since $C'C$ is $\perp$ to $D'C'$, and lies in plane PQ, § 475

$\therefore$ $C'C$ is $\perp$ to DC. § 97

Therefore one common perpendicular can be drawn.

It remains to be proved that no other can be drawn.

If it were possible that another common perpendicular could be drawn, we might suppose *EA* to be ⊥ to both *AB* and *CD*.

Then *EA* would be ⊥ to *AG*, § 97

and therefore *EA* would be ⊥ to the plane *MN*. § 431

Draw *EE'* ⊥ to *D'C'*.

Then *EE'* is ⊥ to the plane *MN*. § 474

But this is impossible, if *EA* is also ⊥ to the plane *MN*. § 437

Hence the supposition that there is a second common perpendicular, *EA*, leads to an absurdity.

Therefore there can be one common perpendicular, and only one, between *AB* and *CD*. Q.E.D.

488. Corollary. *The common perpendicular between two lines not in the same plane is the shortest line joining them.*

How does *CC'* compare in length with *EE'*? Why?
How does *EE'* compare in length with *EA*?

EXERCISE 80

1. Parallel lines have parallel projections on a plane.

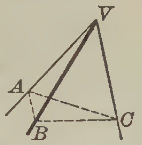
2. If two planes are perpendicular to each other, any line perpendicular to one of them is how related to the other?

3. If three lines passing through a given point *P* are cut by a fourth line that does not pass through *P*, the four lines all lie in the same plane.

4. Seven lines, no three of which lie in the same plane, pass through the same point. How many planes are determined by these lines?

5. A cubical tank 10 in. deep contains water to a depth of 7 in. A foot rule is placed obliquely on the bottom so as just to reach the top edge of the tank. Make a sketch of the tank, and compute the length of the rule covered by water.

489. Polyhedral Angle. The opening of three or more planes which meet at a common point is called a *polyhedral angle.*



The common point V is called the *vertex* of the angle; the intersections VA, VB, etc., of the planes are called the *edges;* the portions of the planes lying between the edges are called the *faces;* and the angles formed by adjacent edges are called the *face angles.*

Every two adjacent edges form a face angle, and every two adjacent faces form a dihedral angle. The face angles and dihedral angles are the *parts* of the polyhedral angle.

490. Size of a Polyhedral Angle. The size of a polyhedral angle depends upon the relative position of its faces, and not upon their extent.

491. Convex and Concave Polyhedral Angles. A polyhedral angle is said to be *convex* or *concave* according as a section made by a plane that cuts all its edges at other points than the vertex is a convex or concave polygon.

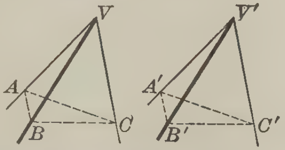
Only convex polyhedral angles are considered in this work.

492. Classes of Polyhedral Angles. A polyhedral angle is called a *trihedral angle* if it has three faces, a *tetrahedral angle* if it has four faces, and so on.

Other names, like pentahedral, hexahedral, heptahedral, etc., for angles with 5, 6, 7, etc., faces, are rarely used.

A polyhedral angle is designated by a letter at the vertex, or by letters representing the vertex and all the faces taken in order. Thus, in the above figure the trihedral angle is designated by V or by $V\text{-}ABC$. A tetrahedral angle would be designated by V or by $V\text{-}ABCD$.

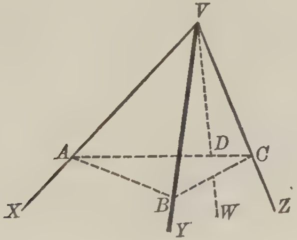
493. Equal Polyhedral Angles. If the corresponding parts of two polyhedral angles are equal and are arranged in the same order, the polyhedral angles are said to be *equal.*



Thus the angles $V\text{-}ABC$ and $V'\text{-}A'B'C'$ are equal. Equal polyhedral angles may evidently be made to coincide by superposition.

PROPOSITION XXV. THEOREM

494. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



Given the trihedral angle *V-XYZ*, with the face angle *XVZ* greater than either of the face angles *XVY* or *YVZ*.

To prove that $\angle XVY + \angle YVZ$ *is greater than* $\angle XVZ$.

Proof. In the $\angle XVZ$ draw VW, making $\angle XVW = \angle XVY$.

Through any point D of VW draw ADC in the plane XVZ.

On VY take VB equal to VD.

Pass a plane through the line AC and the point B.

Then since $AV = AV$, $VD = VB$, and $\angle AVD = \angle AVB$,

$\therefore \triangle AVD$ is congruent to $\triangle AVB$. § 68

$\therefore AD = AB$. § 67

In the $\triangle ABC$, $AB + BC > AC$. § 112

Since $AB = AD$, $\therefore BC > DC$. Ax. 6

In the $\triangle$s BVC and DVC,

$VC = VC$, and $VB = VD$, but $BC > DC$.

$\therefore \angle BVC$ is greater than $\angle DVC$. § 116

$\therefore \angle AVB + \angle BVC$ is greater than $\angle AVD + \angle DVC$. Ax. 6

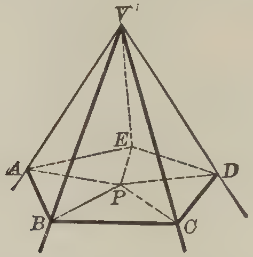
But $\angle AVD + \angle DVC = \angle AVC$. Ax. 11

$\therefore \angle AVB + \angle BVC$ is greater than $\angle AVC$. Ax. 9

That is, $\angle XVY + \angle YVZ$ is greater than $\angle XVZ$. Q.E.D.

PROPOSITION XXVI. THEOREM

495. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Given a convex polyhedral angle V, all of its edges being cut by a plane making the section $ABCDE$.

To prove that $\angle AVB + \angle BVC$, *etc., is less than four rt.* $\angle$s.

Proof. From any point P within the polygon draw PA, PB, PC, PD, PE.

The number of the $\triangle$s having the common vertex P is the same as the number having the common vertex V.

Therefore the sum of the $\angle$s of all the $\triangle$s having the common vertex V is equal to the sum of the $\angle$s of all the $\triangle$s having the common vertex P.

But in the trihedral $\angle$s formed at A, B, C, etc.,

$\angle EAV + \angle BAV$ is greater than $\angle BAE$,

$\angle VBA + \angle CBV$ is greater than $\angle CBA$, etc. § 494

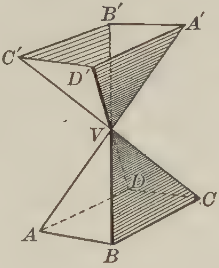
Hence the sum of the $\angle$s at the bases of the $\triangle$s whose common vertex is V is greater than the sum of the $\angle$s at the bases of the $\triangle$s whose common vertex is P. Ax. 7

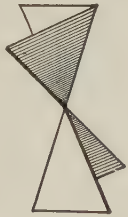
Therefore the sum of the $\angle$s at the vertex V is less than the sum of the $\angle$s at the vertex P. Ax. 7

But the sum of the $\angle$s at P is equal to 4 rt. $\angle$s. § 41

Therefore the sum of the $\angle$s at V is less than 4 rt. $\angle$s. Q. E. D.

496. Symmetric Polyhedral Angles. If the faces of a polyhedral angle $V\text{-}ABCD$ are produced through the vertex V, another polyhedral angle $V\text{-}A'B'C'D'$ is formed, *symmetric* with respect to $\angle V\text{-}ABCD$.





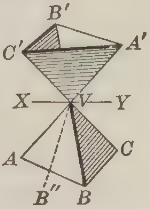
The face angles AVB, BVC, etc., are equal respectively to the face angles $A'VB'$, $B'VC'$, etc. (§ 60).

Also the dihedral angles VA, VB, etc., are equal respectively to the dihedral angles VA', VB', etc. (§ 470). (The second figure shows a pair of these vertical dihedral angles.)

Looked at from the point V, the edges of $\angle V\text{-}ABCD$ are arranged from left to right (counterclockwise) in the order VA, VB, VC, VD, but the edges of $\angle V\text{-}A'B'C'D'$ are arranged from right to left (clockwise) in the order VA', VB', VC', VD'; that is, in an order the reverse of the order of the edges in $\angle V\text{-}ABCD$. Therefore,

Two symmetric polyhedral angles have all their parts equal each to each but arranged in reverse order.

497. Symmetric Polyhedral Angles not Superposable. In general, two symmetric polyhedral angles are not superposable.

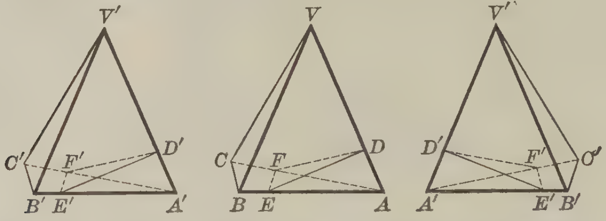


Thus, if the trihedral angle $V\text{-}A'B'C'$ is made to turn 180° about XY, the bisector of the angle CVA', then VA' will coincide with VC, VC' with VA, and the face $A'VC'$ with AVC; but the dihedral angle VA, and hence the dihedral angle VA', not being equal to VC, the plane $A'VB'$ will not coincide with BVC; and, for a similar reason, the plane $C'VB'$ will not coincide with AVB. Hence the edge VB' takes some position VB'' not coincident with VB; that is, the trihedral angles are not superposable.

An analogous case is seen in a pair of gloves. All the parts of one are equal to the corresponding parts of the other, but the right-hand glove will not fit the left hand.

PROPOSITION XXVII. THEOREM

498. *Two trihedral angles are equal or symmetric when the three face angles of the one are equal respectively to the three face angles of the other.*



Given the trihedral angles V and V', the angles BVA, CVA, CVB being equal respectively to the angles $B'V'A'$, $C'V'A'$, $C'V'B'$.

To prove that the angles V and V' are equal or symmetric.

Proof. On the edges of these angles take the six equal segments VA, VB, VC, $V'A'$, $V'B'$, $V'C'$.

Draw AB, BC, CA, $A'B'$, $B'C'$, $C'A'$.

The isosceles $\triangle$ BAV, CAV, CBV are congruent respectively to the isosceles $\triangle$ $B'A'V'$, $C'A'V'$, $C'B'V'$. § 68

$\therefore AB$, BC, CA are equal respectively to $A'B'$, $B'C'$, $C'A'$. § 67

$\therefore \triangle BAC$ is congruent to $\triangle B'A'C'$. § 80

From any point D in VA draw DE in the face AVB and DF in the face AVC, each $\perp$ to VA.

These lines meet AB and AC respectively.

(*For the $\angle$ VAB and VAC are acute, each being one of the equal $\angle$ of an isosceles $\triangle$.*)

Draw EF.

On $A'V'$ take $A'D'$ equal to AD.

Draw $D'E'$ in the face $A'V'B'$ and $D'F'$ in the face $A'V'C'$, each $\perp$ to $V'A'$, and draw $E'F'$.

Then since $AD = A'D'$, Const.

and $\angle DAE = \angle D'A'E'$, § 67

$\therefore$ rt. $\triangle ADE$ is congruent to rt. $\triangle A'D'E'$. § 72

$\therefore AE = A'E'$, and $DE = D'E'$. § 67

In like manner $AF = A'F'$, and $DF = D'F'$.

Furthermore, since it has been proved that

$\triangle BAC$ is congruent to $\triangle B'A'C'$,

$\therefore \angle CAB = \angle C'A'B'$. § 67

$\therefore \triangle AFE$ is congruent to $\triangle A'F'E'$. § 68

$\therefore EF = E'F'$. § 67

$\therefore \triangle EDF$ is congruent to $\triangle E'D'F'$. § 80

$\therefore \angle FDE = \angle F'D'E'$. § 67

$\therefore$ dihedral $\angle VA =$ dihedral $\angle V'A'$. § 473

(*For* $\angle$s *FDE and F'D'E', the measures of these dihedral* $\angle$s, *are equal.*)

In like manner it may be proved that the dihedral angles VB and VC are equal respectively to the dihedral angles $V'B'$ and $V'C'$.

$\therefore$ the trihedral angles V and V' are equal, § 493

or else they are symmetric, by § 496. Q.E.D.

This demonstration applies to either of the two figures denoted by V'-$A'B'C'$, which are symmetric with respect to each other. If the first of these figures is taken, V and V' are equal. If the second is taken, V and V' are symmetric.

499. Corollary. *If two trihedral angles have the three face angles of the one equal respectively to the three face angles of the other, then the dihedral angles of the one are equal respectively to the dihedral angles of the other.*

For whether the trihedral angles are equal or symmetric, as stated in the proposition, the dihedral angles are equal (§§ 493, 496).

EXERCISE 81

1. Find the locus of a point in a space of three dimensions equidistant from two given intersecting lines.

2. Find a point at equal distances from four points not all in the same plane.

3. Two dihedral angles which have their edges parallel and their faces perpendicular are equal or supplementary.

4. The projections on a plane of equal and parallel line-segments are equal and parallel.

5. Two trihedral angles are equal when two dihedral angles and the included face angle of the one are equal respectively to two dihedral angles and the included face angle of the other, and are similarly placed.

6. Two trihedral angles are equal when two face angles and the included dihedral angle of the one are equal respectively to two face angles and the included dihedral angle of the other, and are similarly placed.

7. If the face angle AVB of the trihedral angle V-ABC is bisected by the line VD, the angle CVD is less than, equal to, or greater than half the sum of the angles AVC and BVC, according as $\angle CVD$ is less than, equal to, or greater than 90°.

8. If two face angles of a trihedral angle are equal, the dihedral angles opposite them are equal.

9. A trihedral angle having two of its face angles equal is superposable on its symmetric trihedral angle.

10. Find the locus of a point equidistant from the three edges of a trihedral angle.

11. Find the locus of a point equidistant from the three faces of a trihedral angle.

12. The planes that bisect the dihedral angles of a trihedral angle meet in a straight line.

EXERCISE 82

PROBLEMS OF COMPUTATION

1. From a point P, 4 in. from a plane, a line PX is drawn meeting the plane at X. If PX is 5 in., what is the length of the locus of X in the plane?

2. From a point P, 5 in. from a plane, a line PX is drawn meeting the plane at X. If PX is 12 in., what area is inclosed in the plane by the locus of X? Answer to two decimal places.

3. The base AB of the isosceles triangle ABC in the plane MN is 6 in., and the perimeter of the triangle is 20 in. If the triangle revolves about its base as an axis, what is the greatest distance from the plane that is reached by C? Answer to three decimal places.

4. Two points A and B are 4 in. apart. A point P moves so as to be constantly 5 in. from each of these points. Find the length of the locus of P. Answer to three decimal places.

5. Two parallel planes MN and PQ are cut by a third plane RS so as to make one of the dihedral angles $27° 15' 30''$. Find the other dihedral angles.

6. Two lines are cut by three parallel planes. The segments cut from one line are 3 in. and $5\frac{1}{2}$ in., and those cut from the other line are $7\frac{3}{4}$ in. and x. Find the value of x.

7. Two given planes are at right angles to each other. A point X is 8 in. from each plane. How far is X from the edge of the right dihedral angle?

8. What is the length of the projection on a plane of a line whose length is $10\sqrt{2}$, the inclination of the line to the plane being 45°?

9. From the external point P a perpendicular PP', 9 in. long, is drawn to a plane MN. From P the line PQ is drawn to the plane making the angle $P'PQ$ equal to 30°. Find the length of the projection of PQ on the plane MN.

EXERCISE 83

Review Questions

1. How many and what conditions determine a straight line? How many and what conditions determine a plane?

2. What simple numerical test, following the measurement of certain lengths, determines whether or not one line is perpendicular to another? a line is perpendicular to a plane?

3. How many planes can be passed through a given line perpendicular to a given plane? Is this true for all positions of the given line?

4. Through a given point how many lines can be drawn parallel to a given line? parallel to a given plane? Through a given point how many planes can be passed parallel to a given line? parallel to a given plane?

5. What is the locus, in a line, of a point equidistant from two given points? in a plane? in a space of three dimensions?

6. What is the locus, in a plane, of a point equidistant from two intersecting lines? State a corresponding proposition for solid geometry.

7. What may be said of two lines in one plane perpendicular to the same line? State two corresponding propositions for solid geometry. Does one of these propositions state that two planes perpendicular to the same plane are parallel?

8. What may be said of a line perpendicular to one of two parallel lines? State two corresponding propositions for solid geometry. Is a plane perpendicular to one of two parallel planes perpendicular to the other?

9. If a line is perpendicular to a plane, what may be said of every plane passed through this line? Does a true proposition result from changing the word "perpendicular" to "parallel" in this statement?

BOOK VII

POLYHEDRONS, CYLINDERS, AND CONES

500. Polyhedron. A solid bounded by planes is called a *polyhedron.*

For example, the figures on pages 317 and 318 are polyhedrons.

The bounding planes are called the *faces* of the polyhedron, the intersections of the faces are called the *edges* of the polyhedron, and the intersections of the edges are called the *vertices* of the polyhedron.

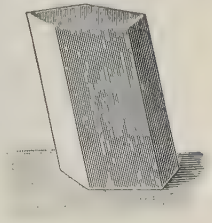
A line joining any two vertices not in the same face is called a *diagonal* of the polyhedron.

501. Section of a Polyhedron. If a plane passes through a polyhedron, the intersection of the plane with such faces as it cuts is called a *section* of the polyhedron.

502. Convex Polyhedron. If every section of a polyhedron is a convex polygon, the polyhedron is said to be *convex.*

Only convex polyhedrons are considered in this work.

503. Prism. A polyhedron of which two faces are congruent polygons in parallel planes, and the other faces are parallelograms having two of their sides in the two parallel planes, is called a *prism.*



The parallel polygons are called the *bases* of the prism, the parallelograms are called the *lateral faces*, and the intersections of the lateral faces are called the *lateral edges.*

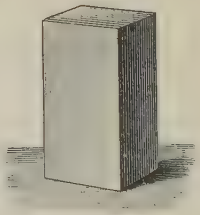
The sum of the areas of the lateral faces is called the *lateral area* of the prism.

The lateral edges of a prism are equal (§ 125).

504. Altitude of a Prism. The perpendicular distance between the planes of the bases of a prism is called its *altitude*

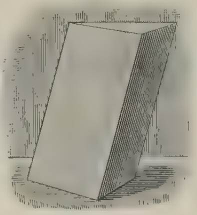
505. Right Prism. A prism whose lateral edges are perpendicular to its bases is called a *right prism*.

The lateral edges of a right prism are equal to the altitude (§ 455).


Right Prism

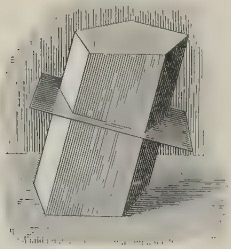
506. Oblique Prism. A prism whose lateral edges are oblique to its bases is called an *oblique prism*.

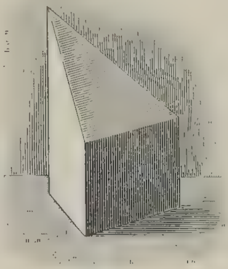
507. Prisms classified as to Bases. Prisms are said to be *triangular*, *quadrangular*, and so on, according as their bases are triangles, quadrilaterals, and so on.


Oblique Triangular Prism

508. Right Section. A section of a prism made by a plane cutting all the lateral edges and perpendicular to them is called a *right section*.

In the case of oblique prisms it is sometimes necessary to produce some of the edges in order that the cutting plane may intersect them.

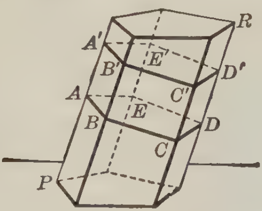

Right Section of a Prism


Truncated Prism

509. Truncated Prism. The part of a prism included between the base and a section made by a plane oblique to the base is called a *truncated prism*.

PROPOSITION I. THEOREM

510. *The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.*



Given the prism *PR* and the parallel sections *AD*, *A'D'* cutting all the lateral edges.

To prove that AD *is congruent to* $A'D'$.

Proof. AB is ∥ to $A'B'$, BC is ∥ to $B'C'$, CD is ∥ to $C'D'$,

and so on for all the corresponding sides. § 453

$\therefore AB = A'B'$, $BC = B'C'$, $CD = C'D'$,

and so on for all the corresponding sides, § 127

and $\angle CBA = \angle C'B'A'$, $\angle DCB = \angle D'C'B'$,

and so on for all the corresponding angles. § 461

$\therefore AD$ is congruent to $A'D'$, by § 142. Q.E.D.

Discussion. Is the proof the same whether or not the two parallel planes are parallel to the bases?

If the sections are all parallel to the bases, are they also congruent to the bases?

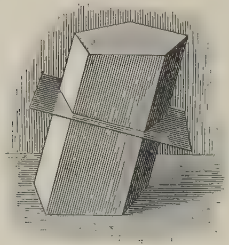
Would the proposition be true if the prism were concave instead of convex?

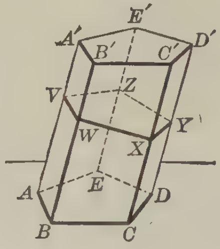
Suppose the bases were squares, what would be known as to the form of the sections?

511. COROLLARY. *Every section of a prism made by a plane parallel to the base is congruent to the base; and all right sections of a prism are congruent.*

PROPOSITION II. THEOREM

512. *The lateral area of a prism is equal to the product of a lateral edge by the perimeter of a right section.*





Given $VWXYZ$ a right section of the prism AD', l the lateral area, e a lateral edge, and p the perimeter of the right section.

To prove that $l = ep$.

Proof. $AA' = BB' = CC' = DD' = EE' = e$. § 503

Furthermore, VW is ⊥ to BB', WX to CC', XY to DD', YZ to EE', and ZV to AA'. § 508

∴ the area of ▱ $AB' = BB' \times VW = e \times VW$, § 322

the area of ▱ $BC' = CC' \times WX = e \times WX$,

the area of ▱ $CD' = DD' \times XY = e \times XY$, and so on.

But l is equal to the sum of these parallelograms. § 503

$\therefore l = e(VW + WX + XY + YZ + ZV)$. Ax. 1

But $VW + WX + XY + YZ + ZV = p$. Ax. 11

$\therefore l = ep$, by Ax. 9. Q.E.D.

513. COROLLARY. *The lateral area of a right prism is equal to the product of the altitude by the perimeter of the base.*

For how would p then compare with $AB + BC + CD + DE + EA$?

The truth of the corollary is easily seen by imagining the right prism laid on one of its lateral faces and the surface as it were unrolled.

EXERCISE 84

Find the lateral areas of the right prisms whose altitudes and perimeters of bases are as follows:

1. $a = 18$ in., $p = 29$ in.

2. $a = 22$ in., $p = 37$ in.

3. $a = 4.25$ in., $p = 6.75$ in.

4. $a = 1$ ft. 7 in., $p = 2$ ft. 9 in.

5. $a = 3$ ft. 8 in., $p = 5$ ft. 7 in.

6. $a = 12$ ft. 2 in., $p = 27$ ft. 9 in.

Find the lateral areas of the prisms whose lateral edges and perimeters of right sections are as follows:

7. $e = 17$ in., $p = 27$ in.

8. $e = 23$ in., $p = 35$ in.

9. $e = 2\frac{3}{4}$ in., $p = 4\frac{7}{8}$ in.

10. $e = 1$ ft. 3 in., $p = 2$ ft. 3 in.

11. $e = 2$ ft. 7 in., $p = 3$ ft. 9 in.

12. $e = 6$ ft. $1\frac{1}{4}$ in., $p = 8$ ft. $9\frac{1}{8}$ in.

Find the lateral edges of the prisms whose lateral areas and perimeters of right sections are as follows:

13. $l = 187$ sq. in., $p = 11$ in.

14. $l = 357$ sq. in., $p = 21$ in.

15. $l = 169$ sq. in., $p = 1$ ft. 1 in.

16. The lateral surface of an iron bar 5 ft. long is to be gilded. The right section is a square whose area is 2.89 sq. in. How many square inches of gilding are required?

17. A right prism of glass is $2\frac{1}{8}$ in. long. Its right section is an equilateral triangle whose altitude is 0.866 in. ($\frac{1}{2}\sqrt{3}$ in.). Find the lateral surface.

18. Find the total area of a right prism whose base is a square with area 5.29 sq. in., and whose length is twice its thickness.

19. What is the total area of a right prism whose altitude is 32 in., and whose base is a right triangle with hypotenuse 106 in. and with one side 84.8 in.?

20. Every section of a prism made by a plane parallel to the lateral edges is a parallelogram.

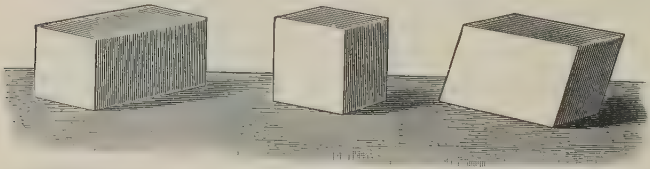
514. Parallelepiped. A prism whose bases are parallelograms is called a *parallelepiped.*

The word is also, with less authority, spelled *parallelopiped.*

515. Right Parallelepiped. A parallelepiped whose edges are perpendicular to the bases is called a *right parallelepiped.*

516. Rectangular Parallelepiped. A right parallelepiped whose bases are rectangles is called a *rectangular parallelepiped.*

By §§ 430 and 453 the four lateral faces are also rectangles.



Rectangular Parallelepiped Cube Oblique Parallelepiped

517. Cube. A parallelepiped whose six faces are all squares is called a *cube.*

We might also say that a hexahedron whose six faces are all squares is a cube, because such a figure would necessarily be a parallelepiped.

518. Unit of Volume. In measuring volumes, a cube whose edges are all equal to the unit of length is taken as the *unit of volume.*

Thus, if we are measuring the contents of a box of which the dimensions are given in feet, we take 1 cubic foot as the unit of volume. If the dimensions are given in inches, we take 1 cubic inch as the unit.

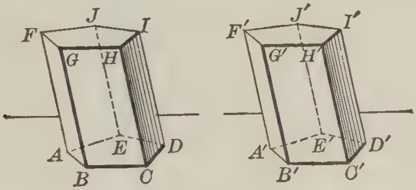
519. Volume. The number of units of volume contained by a solid is called its *volume.*

520. Equivalent Solids. If two solids have equal volumes, they are said to be *equivalent.*

521. Congruent Solids. If two geometric solids are equal in all their parts, and their parts are similarly arranged, the solids are said to be *congruent.*

Proposition III. Theorem

522. *Two prisms are congruent if the three faces which include a trihedral angle of the one are respectively congruent to three faces which include a trihedral angle of the other, and are similarly placed.*



Given the prisms AI and $A'I'$, with the faces AD, AG, AJ respectively congruent to $A'D'$, $A'G'$, $A'J'$, and similarly placed.

To prove that AI *is congruent to* $A'I'$.

Proof. The face $\angle$s BAE, BAF, EAF are equal to the face $\angle$s $B'A'E'$, $B'A'F'$, $E'A'F'$ respectively. § 142

Therefore the trihedral angles A and A' are equal. § 498

Apply the trihedral angle A to its equal A'.

Then the face AD coincides with $A'D'$, AG with $A'G'$, and AJ with $A'J'$; and C falls at C', and D at D'.

The lateral edges of the prisms are parallel. § 446

Therefore CH falls along $C'H'$, and DI along $D'I'$. § 94

Since the points F, G, and J coincide with F', G', and J', each to each, the planes of the upper bases coincide. § 427

Hence H coincides with H', and I with I'.

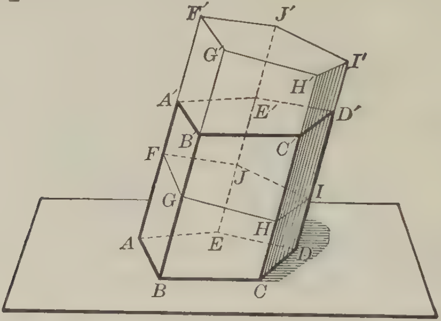
Hence the prisms coincide and are congruent, by § 521. Q. E. D.

523. Corollary 1. *Two truncated prisms are congruent under the conditions given in Proposition III.*

524. Corollary 2. *Two right prisms having congruent bases and equal altitudes are congruent.*

PROPOSITION IV. THEOREM

525. *An oblique prism is equivalent to a right prism whose base is equal to a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Given a right section FI of the oblique prism AD', and FI' a right prism whose lateral edges are equal to the lateral edges of AD'.

To prove that AD' *is equivalent to* FI'.

Proof. If from the equal lateral edges of AD' and FI' we take the lateral edges of FD', which are common to both, the remainders AF and $A'F'$, BG and $B'G'$, etc., are equal. Ax. 2

The bases FI and $F'I'$ are congruent. § 510

Place AI on $A'I'$ so that FI shall coincide with $F'I'$.

Then FA, GB, etc., coincide with $F'A'$, $G'B'$, etc. § 436

Hence the faces GA and $G'A'$, HB and $H'B'$, coincide.

But the faces FI and $F'I'$ coincide.

∴ the truncated prisms AI and $A'I'$ are congruent. § 523

$$\therefore AI + FD' = A'I' + FD'. \qquad \text{Ax. 1}$$

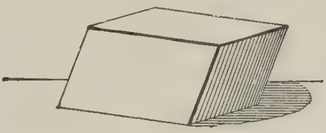
But $$AI + FD' = AD',$$

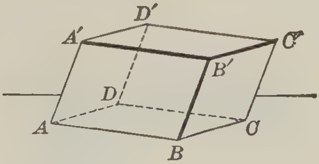
and $$A'I' + FD' = FI'. \qquad \text{Ax. 11}$$

Therefore AD' is equivalent to FI', by Ax. 9. Q. E. D.

PROPOSITION V. THEOREM

526. *The opposite faces of a parallelepiped are congruent and parallel.*





Given a parallelepiped $ABCD\text{-}A'B'C'D'$.

To prove that the opposite faces AB' and DC' are congruent and parallel.

Proof. AB is ∥ to DC, § 118

and $AB = DC$. § 125

Likewise AA' is ∥ and equal to DD'.

$\therefore \angle BAA' = \angle CDD'$. § 461

$\therefore AB'$ is ∥ to DC'. § 461

$\therefore AB'$ is congruent to DC', by § 132. Q.E.D.

EXERCISE 85

1. If in the above figure the three plane angles at A are 80°, 70°, 75°, what are all the other angles in the faces?

2. Given a parallelepiped with the three plane angles at one of the vertices 85°, 75°, 60°, to find all the other angles in the faces.

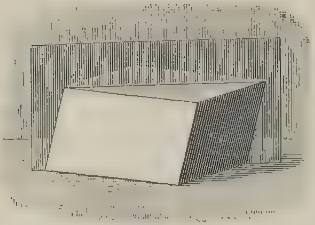
3. Given a rectangular parallelepiped lettered as in the figure above, and with $AB = 4$, $BC = 3$, and $CC' = 3\frac{3}{4}$, to find the length of the diagonal AC'.

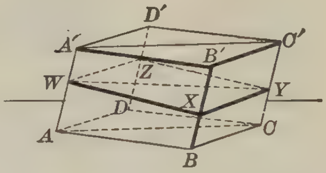
4. The four diagonals of a rectangular parallelepiped are equal.

5. Compute the lengths of the diagonals of a rectangular parallelepiped whose edges from any vertex are a, b, c.

PROPOSITION VI. THEOREM

527. *The plane passed through two diagonally opposite edges of a parallelepiped divides the parallelepiped into two equivalent triangular prisms.*





Given the plane $ACC'A'$ passed through the opposite edges AA' and CC' of the parallelepiped AC'.

To prove that the parallelepiped AC' is divided into two equivalent triangular prisms ABC-B' and ACD-D'.

Proof. Let $WXYZ$ be a right section of the parallelepiped.

The opposite faces AB' and DC' are parallel and equal. § 526

Similarly, the faces AD' and BC' are parallel and equal.

$\therefore WX$ is $\parallel$ to ZY, and WZ to XY. § 453

Therefore $WXYZ$ is a parallelogram. § 118

The plane $ACC'A'$ cuts this parallelogram $WXYZ$ in the diagonal WY. § 429

$\therefore \triangle WXY$ is congruent to $\triangle YZW$. § 126

How shall it be proved that prism ABC-B' is equivalent to a right prism with base WXY and altitude AA'?

How shall it be proved that prism CDA-D' is equivalent to a right prism with base YZW and altitude AA'?

How are these two right prisms known to be equivalent?

How does this prove the proposition?

Discussion. What is the corresponding proposition of plane geometry?

EXERCISE 86

1. The lateral faces of a right prism are rectangles.

2. The diagonals of a parallelepiped bisect one another.

3. The three edges of the trihedral angle at one of the vertices of a rectangular parallelepiped are 5 in., 6 in., and 7 in. respectively. Required the total area of the six faces of the parallelepiped.

4. The three face angles at one vertex of a parallelepiped are each 60°, and the three edges of the trihedral angle with that vertex are 3 in., 2 in., 1 in. respectively. Required the total area of the six faces. Answer to two decimal places.

5. In a rectangular parallelepiped the square on any diagonal is equivalent to the sum of the squares on any three edges that meet at one of the vertices.

6. In a box 3 in. deep and 6 in. wide a wire 1 ft. long can be stretched to reach from one corner to the diagonally opposite corner. Required the length of the box. Answer to two decimal places.

7. The diagonal of the base of a rectangular parallelepiped is $31\frac{3}{5}$ in. and the height of the parallelepiped is 23.7 in. Required the length of the diagonal of the parallelepiped.

8. The total area of the six faces of a cube is 18 sq. in. Find the diagonal of the cube.

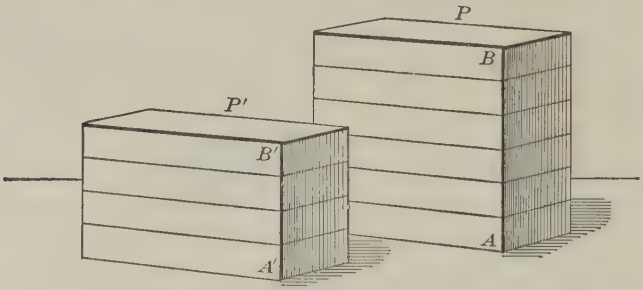
9. The diagonal of the face of a cube equals $\sqrt{14}$. Find the diagonal of the cube.

10. The diagonal of a cube equals $2.75\sqrt{3}$. Find the diagonal of a face of the cube.

11. A water tank is 3 ft. long, 2 ft. 6 in. wide, and 1 ft. 9 in. deep. How many square feet of zinc will be required to line the four sides and the base, allowing $1\frac{7}{8}$ sq. ft. for overlapping and for turning the top edge?

PROPOSITION VII. THEOREM

528. *Two rectangular parallelepipeds having congruent bases are to each other as their altitudes.*



Given two rectangular parallelepipeds P and P', with congruent bases and with altitudes AB and $A'B'$.

To prove that $P : P' = AB : A'B'$.

CASE 1. *When AB and $A'B'$ are commensurable.*

Proof. Suppose a common measure of AB and $A'B'$ to be contained m times in AB, and n times in $A'B'$.

Then $AB : A'B' = m : n$.

Apply this measure to AB and $A'B'$, and through the several points of division pass planes perpendicular to these lines.

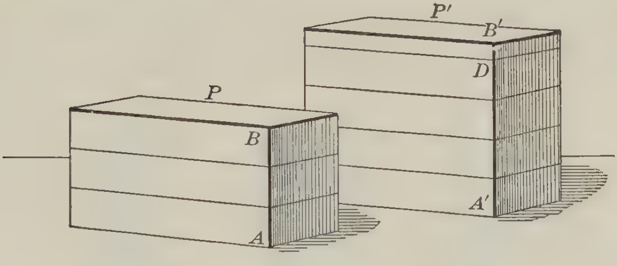
These planes divide the parallelepiped P into m parallelepipeds and the parallelepiped P' into n parallelepipeds, congruent each to each. § 524

$\therefore P : P' = m : n.$

$\therefore P : P' = AB : A'B'$, by Ax. 8. Q.E.D.

The proof for the incommensurable case is similar to that in other propositions of this nature. It may be omitted at the discretion of the teacher without destroying the sequence, if the incommensurable cases are not being considered by the class.

Case 2. *When AB and A'B' are incommensurable.*



Proof. Divide AB into any number of equal parts, and apply one of these parts to $A'B'$ as a unit of measure as many times as $A'B'$ will contain it.

Since AB and $A'B'$ are incommensurable, a certain number of these parts will extend from A' to a point D, leaving a remainder DB' less than one of the parts.

Through D pass a plane $\perp$ to $A'B'$, and let Q denote the parallelepiped whose base is the same as that of P', and whose altitude is $A'D$.

Then $Q : P = A'D : AB$. Case 1

If the number of parts into which AB is divided is indefinitely increased, the ratio $Q : P$ approaches $P' : P$ as a limit, and the ratio $A'D : AB$ approaches $A'B' : AB$ as a limit. § 204

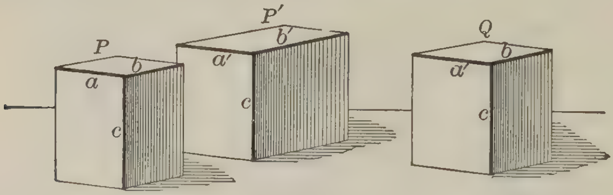
The remainder of the proof of the incommensurable case is substantially as in the proof given on page 297, and it is therefore left for the student.

529. Dimensions. The lengths of the three edges of a rectangular parallelepiped which meet at a common vertex are called its *dimensions.*

530. Corollary. *Two rectangular parallelepipeds which have two dimensions in common are to each other as their third dimensions.*

PROPOSITION VIII. THEOREM

531. *Two rectangular parallelepipeds having equal altitudes are to each other as their bases.*



Given two rectangular parallelepipeds, P and P', and a, b, c, and a', b', c, their three dimensions respectively.

To prove that $$\frac{P}{P'}=\frac{ab}{a'b'}.$$

Proof. Let Q be a third rectangular parallelepiped whose dimensions are a', b, and c.

Now Q has the two dimensions b and c in common with P, and the two dimensions a' and c in common with P'.

Therefore $$\frac{P}{Q}=\frac{a}{a'},$$

and $$\frac{Q}{P'}=\frac{b}{b'}. \qquad \text{§ 530}$$

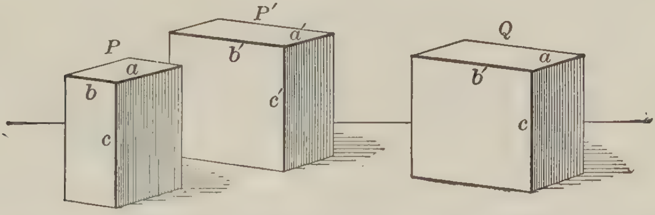
The products of the corresponding members of these two equations give $$\frac{P}{P'}=\frac{ab}{a'b'}, \text{ by Ax. 3.}$$ Q.E.D.

532. COROLLARY. *Two rectangular parallelepipeds which have one dimension in common are to each other as the products of their other two dimensions.*

For any edge of a rectangular parallelepiped may be taken as the altitude, whence § 531 applies.

Proposition IX. Theorem

533. *Two rectangular parallelepipeds are to each other as the products of their three dimensions.*



Given two rectangular parallelepipeds, P and P', and a, b, c, and a', b', c', their three dimensions respectively.

To prove that $\frac{P}{P'} = \frac{abc}{a'b'c'}$.

Proof. Let Q be a third rectangular parallelepiped whose dimensions are a, b', and c.

Then $\frac{P}{Q} = \frac{b}{b'}$, § 530

and $\frac{Q}{P'} = \frac{ac}{a'c'}$. § 532

$\therefore \frac{P}{P'} = \frac{abc}{a'b'c'}$, by Ax. 3. Q.E.D.

534. Corollary 1. *The volume of a rectangular parallelepiped is equal to the product of its three dimensions.*

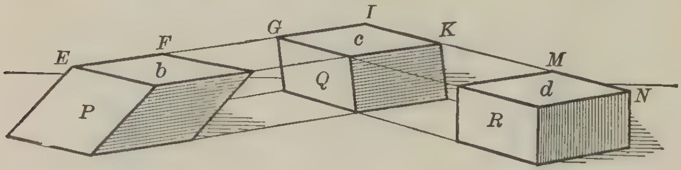
For in the above case, if $a' = b' = c' = 1$, then $P' = 1 \times 1 \times 1 = 1$ (§ 518). But the volume of P (§ 519) is $P : P'$, and $P : P' = abc : 1$ (§ 533). Therefore the volume of P is abc.

535. Corollary 2. *The volume of a rectangular parallelepiped is equal to the product of its base and altitude.*

For the volume of P is abc, and ab equals the base and c the altitude.

PROPOSITION X. THEOREM

536. *The volume of any parallelepiped is equal to the product of its base by its altitude.*



Given an oblique parallelepiped P of volume v, with no two of its faces perpendicular, with base b and with altitude a.

To prove that $v = ba$.

Proof. Produce the edge EF and the edges ∥ to EF, and cut them perpendicularly by two parallel planes whose distance apart GI is equal to EF. We then have the oblique parallelepiped Q whose base c is a rectangle.

Produce the edge IK and the edges ∥ to IK, and cut them perpendicularly by two planes whose distance apart MN is equal to IK. We then have the rectangular parallelepiped R.

Now $P = Q$, and $Q = R$. § 525

$\therefore P = R$. Ax. 8

The three parallelepipeds have a common altitude a. § 455

Also $b = c$, § 323

and $c = d$. § 133

$\therefore b = d$. Ax. 8

But the volume of $R = da$. § 535

Putting P for R, and b for d, we have $v = ba$, by Ax. 9. Q.E.D.

537. COROLLARY. *The volume of any parallelepiped is equal to that of a rectangular parallelepiped of equivalent base and equal altitude.*

EXERCISE 87

1. Find the ratio of two rectangular parallelepipeds, if their dimensions are 3, 4, 5, and 9, 8, 10 respectively.

2. Find the ratio of two rectangular parallelepipeds, if their altitudes are each 6 in., and their bases 5 in. by 4 in., and 10 in. by 8 in. respectively.

3. Find the volume of a rectangular parallelepiped 2 ft. 6 in. long, 1 ft. 8 in. wide, and 1 ft. 6 in. high.

4. Find the volume of a rectangular parallelepiped whose base is 27 sq. in. and whose altitude is $13\frac{1}{2}$ in.

5. The volume of a rectangular parallelepiped is 1152 cu. in. and the area of the base is half a square foot. Find the altitude.

6. The volume of a rectangular parallelepiped with a square base is 273.8 cu. in. and the altitude is 5 in. Find the dimensions.

7. A rectangular tank full of water is 7 ft. 3 in. long by 4 ft. 6 in. wide. How many cubic feet of water must be drawn off in order that the surface may be lowered a foot?

8. Find to two decimal places the length of each side of a cubic reservoir that will contain exactly a gallon (231 cu. in.).

9. A box has as its internal dimensions 18 in., $9\frac{1}{2}$ in., and $4\frac{1}{4}$ in. The box and cover are made of steel $\frac{1}{8}$ in. thick. If steel weighs 490 lb. per cubic foot, what is the weight of the box?

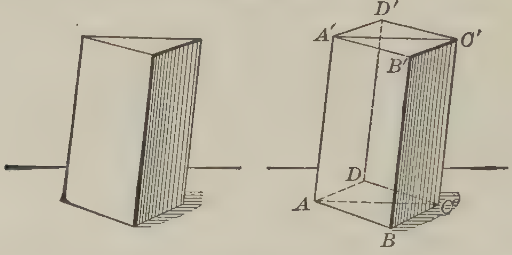
10. A steel rod 4 ft. 8 in. long is 2 in. wide and $1\frac{1}{2}$ in. thick. How much does it weigh, at 490 lb. per cubic foot?

11. If 3 cu. in. of gold beaten into gold leaf will cover 75,000 sq. in. of surface, find the thickness of the leaf.

12. The sum of the squares on the four diagonals of a parallelepiped is equivalent to the sum of the squares on the twelve edges.

PROPOSITION XI. THEOREM

538. *The volume of a triangular prism is equal to the product of its base by its altitude.*



Given the triangular prism $ABC\text{-}B'$, with volume v, base b, and altitude a.

To prove that $v = ba$.

Proof. Upon the edges AB, BC, BB' construct the parallelepiped $ABCD\text{-}B'$.

Then $ABC\text{-}B' = \frac{1}{2}\,ABCD\text{-}B'$. § 527

The volume of $ABCD\text{-}B' = ABCD \times a$. § 536

But $ABCD = 2\,b$. § 126

$\therefore v = \frac{1}{2}(2\,ba) = ba$, by Ax. 9. Q.E.D.

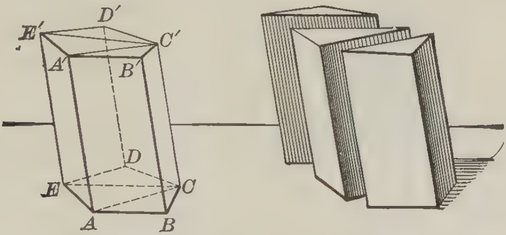
EXERCISE 88

Find the volumes of the triangular prisms whose bases and altitudes are as follows:

1. 17 sq. in., 8 in.
2. 15.75 sq. ft., 3 ft.
3. 3¼ sq. ft., 1 ft. 8 in.
4. 5½ sq. ft., 2 ft. 9 in.
5. 15.84 sq. ft., 3 ft. 10 in.
6. 16⅔ sq. in., 2⅖ in.
7. 22½ sq. in., 4½ in.
8. 33⅓ sq. in., 7⅕ in.
9. 42⅞ sq. in., 3⅜ in.
10. 27⅗ sq. in., 3¾ in.
11. 12 sq. ft. 75 sq. in., 2 ft. 7 in.

Proposition XII. Theorem

539. *The volume of any prism is equal to the product of its base by its altitude.*



Given the prism AC' with volume v, base b, and altitude a.

To prove that $v = ba$.

Proof. It is possible to divide any prism in general into what kind of simpler prisms?

How is this done?

What is the volume of each of these simpler prisms (§ 538)?

What is the sum of the volumes of these simpler prisms?

What is the sum of their bases?

How does the common altitude of these simpler prisms compare with a, the altitude of the given prism?

What conclusion can be drawn from these statements?

Write the proof in full.

540. Corollary 1. *Prisms having equivalent bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases.*

Write the proof in full.

541. Corollary 2. *Prisms having equivalent bases and equal altitudes are equivalent.*

Write the proof in full.

EXERCISE 89

1. If the length of a rectangular parallelepiped is 18 in., the width 9 in., and the height 8 in., find the total area of the surface.

2. Find the volume of a triangular prism, if its height is 15 in. and the sides of the base are 6 in., 5 in., and 5 in.

3. Find the volume of a prism whose height is 15 ft., if each side of the triangular base is 10 in.

4. The base of a right prism is a rhombus of which one side is 20 in., and the shorter diagonal 24 in. The height of the prism is 30 in. Find the entire surface and the volume.

5. How many square feet of lead will be required to line an open cistern which is 4 ft. 6 in. long, 2 ft. 8 in. wide, and contains 42 cu. ft.?

6. An open cistern 6 ft. long and $4\frac{1}{2}$ ft. wide holds 108 cu. ft. of water. How many square feet of lead will it take to line the sides and bottom?

7. One edge of a cube is e. Find in terms of e the surface, the volume, and the length of a diagonal of the cube.

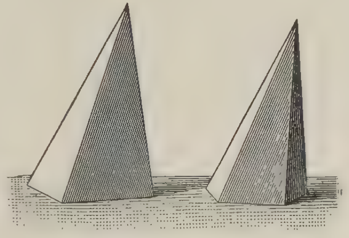
8. The diagonal of one of the faces of a cube is d. Find in terms of d the volume of the cube.

9. The three dimensions of a rectangular parallelepiped are a, b, c. Find in terms of a, b, and c the volume and the area of the surface.

10. Find the volume of a prism with bases regular hexagons, if the height is 10 ft. and each side of the hexagons is 10 in.

11. An open cistern is made of iron $\frac{1}{4}$ in. thick. The inner dimensions are: length, 4 ft. 6 in.; breadth, 3 ft.; depth, 2 ft. 6 in. What will the cistern weigh when empty? when full of water? (A cubic foot of water weighs $62\frac{1}{2}$ lb. Iron is 7.2 times as heavy as water; that is, the specific gravity of iron is 7.2.)

542. Pyramid. A polyhedron of which one face, called the *base*, is a polygon of any number of sides and the other faces are triangles having a common vertex is called a *pyramid*.



The triangular faces having a common vertex are called the *lateral faces*, their intersections are called the *lateral edges*, and their common vertex is called the *vertex* of the pyramid. The base of a pyramid may be any kind of a polygon, but usually a convex polygon is taken.

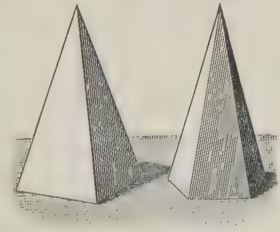
543. Lateral Area. The sum of the areas of the lateral faces of a pyramid is called the *lateral area* of the pyramid.

544. Altitude. The perpendicular distance from the vertex to the plane of the base is called the *altitude* of the pyramid.

545. Pyramids classified as to Bases. Pyramids are said to be *triangular*, *quadrangular*, and so on, according as their bases are triangles, quadrilaterals, and so on.

A triangular pyramid has four triangular faces and is called a *tetrahedron*. Any one of its faces may be taken as the base.

546. Regular Pyramid. If the base of a pyramid is a regular polygon whose center coincides with the foot of the perpendicular let fall from the vertex to the base, the pyramid is called a *regular pyramid*.



A regular pyramid is also called a *right pyramid*.

547. Slant Height of a Regular Pyramid. The altitude of any one of the lateral faces of a regular pyramid, drawn from the vertex of the pyramid, is called the *slant height*.

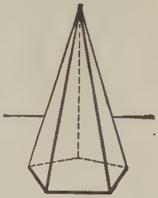
The slant height is the same whatever face is taken (§ 439). Only a regular pyramid can have a slant height.

548. Properties of Regular Pyramids. Among the properties of regular pyramids the following are too evident to require further proof than that referred to below:

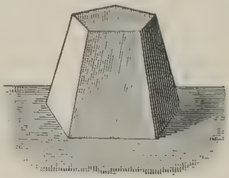
(1) *The lateral edges of a regular pyramid are equal* (§ 439).

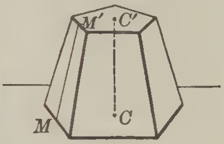
(2) *The lateral faces of a regular pyramid are congruent isosceles triangles* (§ 80).

(3) *The slant height of a regular pyramid is the same for all the lateral faces* (§ 439).



549. Frustum of a Pyramid. The portion of a pyramid included between the base and a section parallel to the base is called a *frustum of a pyramid.*





The base of the pyramid and the parallel section are called the *bases* of the frustum.

A more general term, including frustum as a special case, is *truncated pyramid*, the portion of a pyramid included between the base and *any* section made by a plane that cuts all the lateral edges. This term is little used.

550. Altitude of a Frustum. The perpendicular distance between the bases is called the *altitude* of the frustum.

E.g. $C'C$ is the altitude of the frustum in the above figure.

551. Lateral Faces of a Frustum. The portions of the lateral faces of a pyramid that lie between the bases of a frustum are called the *lateral faces* of the frustum.

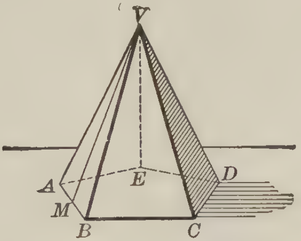
In the case of a frustum of a regular pyramid the lateral faces are congruent isosceles trapezoids. The sum of the areas of the lateral faces is called the *lateral area* of the frustum.

552. Slant Height of a Frustum. The altitude of one of the trapezoid faces of a frustum of a regular pyramid is called the *slant height* of the frustum.

Thus MM' in the above figure is the slant height.

PROPOSITION XIII. THEOREM

553. *The lateral area of a regular pyramid is equal to half the product of its slant height by the perimeter of its base.*



Given the regular pyramid *V-ABCDE*, with *l* the lateral area, *s* the slant height, and *p* the perimeter of the base.

To prove that $l = \frac{1}{2} sp$.

Proof. The $\triangle$ VAB, VBC, VCD, VDE, and VEA are congruent. § 548

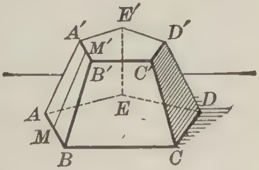
The area of each $\triangle = \frac{1}{2} s \times$ its base. § 325

The sum of the bases of the triangles $= p$. Ax. 11

$\therefore$ the sum of the areas of these $\triangle = \frac{1}{2} sp$. Ax. 1

But the sum of the areas of these $\triangle = l$. § 543

$\therefore l = \frac{1}{2} sp$, by Ax. 8. Q.E.D.



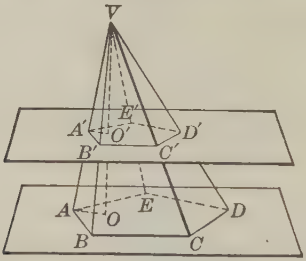
554. COROLLARY. *The lateral area of the frustum of a regular pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height of the frustum.*

How is the area of a trapezoid found (§ 329)? Are these trapezoids congruent? What is the sum of their lower bases? of their upper bases? What is the sum of their areas? Insert the formula.

PROPOSITION XIV. THEOREM

555. *If a pyramid is cut by a plane parallel to the base:*

1. *The edges and altitude are divided proportionally.*
2. *The section is a polygon similar to the base.*



Given the pyramid *V-ABCDE* cut by a plane parallel to its base, intersecting the lateral edges in *A′*, *B′*, *C′*, *D′*, *E′*, and the altitude *VO* in *O′*.

1. *To prove that* $\frac{VA'}{VA} = \frac{VB'}{VB} = \cdots = \frac{VO'}{VO}$.

Proof. Since the plane $A'D'$ is ∥ to the plane AD, Given

∴ $A'B'$ is ∥ to AB, $B'C'$ is ∥ to BC, $\cdots$, and $A'O'$ is ∥ to AO. § 453

$$\therefore \frac{VA'}{VA} = \frac{VB'}{VB} = \cdots = \frac{VO'}{VO}, \text{ by § 274.}$$ Q.E.D.

2. *To prove the section $A'B'C'D'E'$ similar to the base $ABCDE$.*

Proof. Since $\triangle VA'B'$ is similar to $\triangle VAB$, $\triangle VB'C'$ similar to $\triangle VBC$, and so on (why?), how can the corresponding sides of the polygons be proved proportional?

Since $A'B'$ is ∥ to AB, $B'C'$ to BC, etc. (why?), how can the corresponding angles be proved equal?

Then why is $A'B'C'D'E'$ similar to $ABCDE$?

556. Corollary 1. *Any section of a pyramid parallel to the base is to the base as the square of the distance from the vertex is to the square of the altitude of the pyramid.*

For $$\frac{VO'}{VO} = \frac{VA'}{VA} \qquad \text{§ 555}$$

$$= \frac{A'B'}{AB}. \qquad \text{§ 288}$$

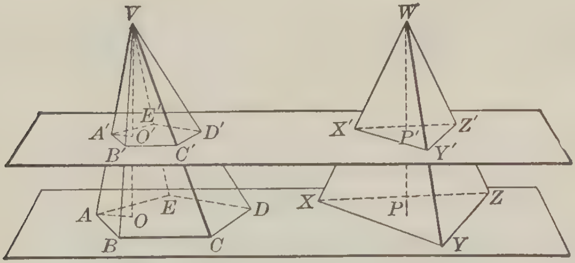
Therefore $$\frac{\overline{VO'}^2}{\overline{VO}^2} = \frac{\overline{A'B'}^2}{\overline{AB}^2}. \qquad \text{§ 270}$$

But, from similar polygons,

$$\frac{A'B'C'D'E'}{ABCDE} = \frac{\overline{A'B'}^2}{\overline{AB}^2}. \qquad \text{§ 334}$$

Hence, by substituting,

$$\frac{A'B'C'D'E'}{ABCDE} = \frac{\overline{VO'}^2}{\overline{VO}^2}. \qquad \text{Ax. 8}$$

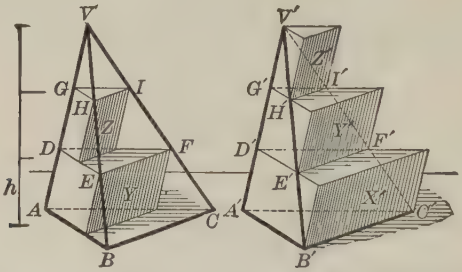


557. Corollary 2. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the bases, and at equal distances from the vertices, are equivalent.*

What is the ratio of $A'B'C'D'E'$ to $ABCDE$?
How can this be shown to equal $\overline{VO'}^2 : \overline{VO}^2$?
What is the ratio of $X'Y'Z'$ to XYZ?
How can this be shown to equal $\overline{WP'}^2 : \overline{WP}^2$?
Are the ratios $\overline{VO'}^2 : \overline{VO}^2$ and $\overline{WP'}^2 : \overline{WP}^2$ equal?
Since it is given that $ABCDE = XYZ$ what can be said of $A'B'C'D'E'$ and $X'Y'Z'$?

Proposition XV. Theorem

558. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Given two triangular pyramids, *V-ABC* and *V′-A′B′C′*, having equivalent bases and equal altitudes.

To prove that V-ABC and V′-A′B′C′ are equivalent.

Proof. Suppose the pyramids are not equivalent, and

$$V'\text{-}A'B'C' > V\text{-}ABC.$$

Place the bases in the same plane, and suppose the altitude divided into n equal parts, calling each of these parts h.

Through the points of division pass planes parallel to the base, cutting the pyramids in DEF, GHI, $\cdots$, $D'E'F'$, $G'H'I'$, $\cdots$.

On $A'B'C'$, $D'E'F'$, $G'H'I'$, and other parallel sections, if any, construct prisms with lateral edges parallel to $A'V'$, and with altitude h. In the figure these are represented by X', Y', and Z'.

On DEF, GHI, and other parallel sections, if any, as upper bases, construct the prisms Y, Z, with lateral edges parallel to VA, and with altitude h.

Then since $DEF = D'E'F'$, § 557

and $h = h$, Iden.

$\therefore$ prism $Y =$ prism Y'. § 541

Similarly prism $Z =$ prism Z'.

But $$X' + Y' + Z' > V'\text{-}A'B'C',$$

and $$Y + Z < V\text{-}ABC. \qquad \text{Ax. 11}$$

$$\therefore V'\text{-}A'B'C' - V\text{-}ABC < X' + Y' + Z' - (Y + Z),$$

or $$V'\text{-}A'B'C' - V\text{-}ABC < X'.$$

That is, the difference between the pyramids must be less than the difference between the sets of prisms.

Now by increasing n indefinitely, and consequently decreasing h indefinitely, X' can be made less than any assigned quantity.

Hence whatever difference we suppose to exist between the pyramids, X' can be made smaller than that supposed difference.

But this is absurd, since we have shown that X' is greater than the difference, if any exists.

Hence it leads to a manifest absurdity to suppose that $V'\text{-}A'B'C' > V\text{-}ABC$.

In the same way it leads to an absurdity to suppose that $V\text{-}ABC > V'\text{-}A'B'C'$.

$$\therefore V\text{-}ABC = V'\text{-}A'B'C'. \qquad \text{Q.E.D.}$$

EXERCISE 90

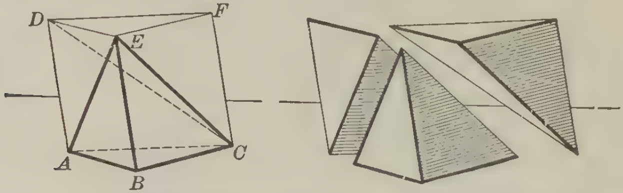
1. The slant height of a regular pyramid is 6 in., and the base is an equilateral triangle of altitude $2\sqrt{3}$ in. Find the lateral area of the pyramid.

2. The slant height of a regular triangular pyramid equals the altitude of the base. The area of the base is $\sqrt{3}$ sq. ft. Find the total area of the pyramid.

3. A pyramid has for its base a right triangle with hypotenuse 5 and shortest side 3. Another one of equal altitude has for its base an equilateral triangle with side $2\sqrt{2\sqrt{3}}$. Prove the pyramids equivalent.

PROPOSITION XVI. THEOREM

559. *The volume of a triangular pyramid is equal to one third the product of its base by its altitude.*



Given the triangular pyramid $E\text{-}ABC$**, with volume** v**, base** b**, and altitude** a**.**

To prove that $v = \frac{1}{3}ba$.

Proof. On the base ABC construct a prism $ABC\text{-}DEF$.

Through DE and EC pass a plane CDE.

Then the prism is composed of three triangular pyramids $E\text{-}ABC$, $E\text{-}CFD$, and $E\text{-}ACD$.

Now the pyramids $E\text{-}CFD$ and $E\text{-}ACD$ have the same altitude and equal bases CFD and ACD. § 126

$\therefore E\text{-}CFD = E\text{-}ACD$. § 558

But pyramid $E\text{-}CFD$ is the same as pyramid $C\text{-}DEF$, which has the same altitude as pyramid $E\text{-}ABC$, and has base DEF equal to base ABC. § 511

$\therefore E\text{-}CFD = E\text{-}ABC$. § 558

$\therefore E\text{-}ABC = E\text{-}CFD = E\text{-}ACD$. Ax. 8

$\therefore$ pyramid $E\text{-}ABC = \frac{1}{3}$ prism $ABC\text{-}DEF$.

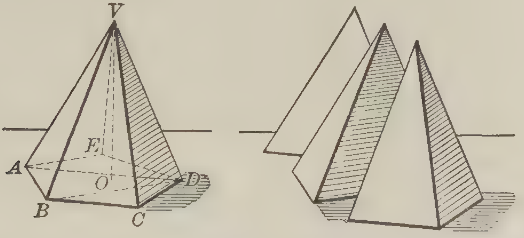
But the volume of $ABC\text{-}DEF = ba$. § 539

$\therefore v = \frac{1}{3}ba$, by Ax. 4. Q.E.D.

560. COROLLARY. *The volume of a triangular pyramid is equal to one third the volume of a triangular prism of the same base and altitude.*

Proposition XVII. Theorem

561. *The volume of any pyramid is equal to one third the product of its base by its altitude.*



Given the pyramid $V\text{-}ABCDE$, with volume v, base b, and altitude a.

To prove that $v = \frac{1}{3} ba$.

Proof. Through the edge VD and the diagonals of the base, DA, DB, pass planes.

These planes divide the pyramid $V\text{-}ABCDE$ into three triangular pyramids.

What can be said as to the altitudes of the original pyramid and of the triangular pyramids?

What can be said as to the base of the original pyramid in relation to the bases of the triangular pyramids?

What is the volume of each triangular pyramid?

What is the sum of the volumes of the triangular pyramids?

Complete the proof.

562. Corollary. *The volumes of two pyramids are to each other as the products of their bases and altitudes; pyramids having equivalent bases are to each other as their altitudes; pyramids having equal altitudes are to each other as their bases; pyramids having equivalent bases and equal altitudes are equivalent.*

EXERCISE 91

Find the lateral areas of regular pyramids, given the slant heights and the perimeters of the bases, as follows:

1. $s = 34$ in., $p = 57$ in. **3.** $s = 2$ ft. 7 in., $p = 4$ ft. 6 in.

2. $s = 8\frac{3}{4}$ in., $p = 17\frac{1}{8}$ in. **4.** $s = 127$ ft. 5 in., $p = 63$ ft. 2 in.

Find the lateral areas of frustums of regular pyramids, given the slant heights of the frustums and the perimeters of the bases, as follows:

5. $s = 4$ in., $p = 8$ in., $p' = 6$ in.

6. $s = 5\frac{1}{8}$ in., $p = 9\frac{3}{4}$ in., $p' = 7\frac{3}{8}$ in.

7. $s = 2$ ft. 3 in., $p = 4$ ft. 8 in., $p' = 3$ ft. 9 in.

Find the volumes of pyramids, given the altitudes and the areas of the bases, as follows:

8. $a = 7$ in., $b = 9$ sq. in. **11.** $a = 3\frac{1}{8}$ in., $b = 5\frac{1}{8}$ sq. in.

9. $a = 6$ in., $b = 23$ sq. in. **12.** $a = 4\frac{3}{8}$ in., $b = 19$ sq. in.

10. $a = 17$ in., $b = 51$ sq. in. **13.** $a = 27.5$ ft., $b = 325$ sq. ft.

Find the lateral areas of regular pyramids, given the slant heights, the number of sides of the bases, and the length of each side, as follows:

14. $s = 2.3$ in., $n = 4$, $l = 2.1$ in.

15. $s = 3.7$ in., $n = 6$, $l = 2.9$ in.

16. $s = 5.33$ in., $n = 8$, $l = 3$ in.

Find the volumes of pyramids, given the altitudes and a description of the bases, as follows:

17. $a = 7$ in., the base a square with side 2 in.

18. $a = 6\frac{3}{4}$ in., the base a square with diagonal $3\sqrt{2}$ in.

19. $a = 8.9$ in., the base a triangle with each side 3.7 in.

20. Find the lateral area of a regular pyramid, if the slant height is 16 ft. and the base is a hexagon with side 12 ft.

21. Find the lateral area of a regular pyramid, if the slant height is 8 ft. and the base is a pentagon with side 5 ft.

22. Find the total surface of a regular pyramid, if the slant height is 6 ft. and the base is a square with side 4 ft.

23. Find the total surface of a regular pyramid, if the slant height is 18 ft. and the base is a square with side 8 ft.

24. Find the total surface of a regular pyramid, if the slant height is 16 ft. and the base is a triangle with side 8 ft.

25. The volume of a pyramid is 26 cu. ft. 936 cu. in. and each side of its square base is 3 ft. 6 in. Find the height.

26. The volume of a pyramid is 20 cu. ft. and the sides of its triangular base are 5 ft., 4 ft., and 3 ft. respectively. Find the height.

27. Find the volume of a regular pyramid with a square base whose side is 40 ft., the lateral edge being 101 ft.

28. Find the volume of a regular pyramid whose slant height is 12 ft. and whose base is an equilateral triangle inscribed in a circle of radius 10 ft.

29. Having given the base edge a and the total surface t of a regular pyramid with a square base, find the height h.

30. Having given the base edge a and the total surface t of a regular pyramid with a square base, find the volume v.

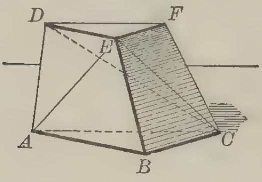
31. The eight edges of a regular pyramid with a square base are equal and the total surface is t. Find the edge.

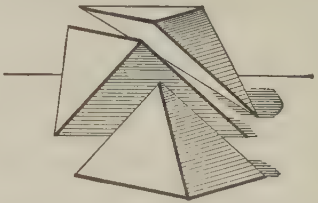
32. Find the base edge a of a regular pyramid with a square base, having given the height h and the total surface t.

33. Show how to find the volume of any polyhedron by dividing the polyhedron into pyramids.

PROPOSITION XVIII. THEOREM

563. *The frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and the mean proportional between the two bases of the frustum.*





Given the frustum of a triangular pyramid, $ABC\text{-}DEF$, having ABC, or b, for its lower base; DEF, or b', for its upper base; and the altitude a.

To prove that $ABC\text{-}DEF = \frac{1}{3}ab + \frac{1}{3}ab' + \frac{1}{3}a\sqrt{bb'}$.

Proof. Through A, E, and C, and also through C, D, and E, pass planes dividing the frustum into three pyramids.

Then $E\text{-}ABC = \frac{1}{3}ab$,

and $C\text{-}DEF = \frac{1}{3}ab'$. § 559

It therefore remains only to prove that $E\text{-}ACD = \frac{1}{3}a\sqrt{bb'}$.

We see by the figure that we may speak of $E\text{-}ABC$ as $C\text{-}ABE$, and of $E\text{-}ACD$ as $C\text{-}AED$.

But $C\text{-}ABE : C\text{-}AED = \triangle ABE : \triangle AED$. § 562

Since $\triangle$ ABE and AED have for a common altitude the altitude of the trapezoid $ABED$,

$\therefore \triangle ABE : \triangle AED = AB : DE$. § 327

$\therefore C\text{-}ABE : C\text{-}AED = AB : DE$, Ax. 8

or $E\text{-}ABC : E\text{-}ACD = AB : DE$. Ax. 9

In like manner $E\text{-}ACD$ and $E\text{-}CFD$ have a common vertex E and have their bases in the same plane, $ACFD$, so that

$$E\text{-}ACD : E\text{-}CFD = \triangle ACD : \triangle CFD. \qquad \S\ 562$$

Since $\triangle$s ACD and CFD have for a common altitude the altitude of the trapezoid $ACFD$,

$$\therefore \triangle ACD : \triangle CFD = AC : DF. \qquad \S\ 327$$

$$\therefore E\text{-}ACD : E\text{-}CFD = AC : DF. \qquad \text{Ax. } 8$$

But $\triangle DEF$ is similar to $\triangle ABC$. § 555

$$\therefore AB : DE = AC : DF. \qquad \S\ 282$$

$$\therefore E\text{-}ABC : E\text{-}ACD = AC : DF. \qquad \text{Ax. } 8$$

$$\therefore E\text{-}ABC : E\text{-}ACD = E\text{-}ACD : E\text{-}CFD. \qquad \text{Ax. } 8$$

But $E\text{-}CFD$ is the same as $C\text{-}DEF$, which has been shown to equal $\frac{1}{3}ab'$.

$$\therefore \tfrac{1}{3}ab : E\text{-}ACD = E\text{-}ACD : \tfrac{1}{3}ab'. \qquad \text{Ax. } 9$$

$$\therefore E\text{-}ACD = \sqrt{\tfrac{1}{3}ab \times \tfrac{1}{3}ab'} \qquad \S\ 262$$

$$= \tfrac{1}{3}a\sqrt{bb'}.$$

$$\therefore E\text{-}ABC + C\text{-}DEF + E\text{-}ACD = \tfrac{1}{3}ab + \tfrac{1}{3}ab' + \tfrac{1}{3}a\sqrt{bb'}. \qquad \text{Ax. } 1$$

That is, $ABC\text{-}DEF = \frac{1}{3}ab + \frac{1}{3}ab' + \frac{1}{3}a\sqrt{bb'}$, by Ax. 9. Q.E.D.

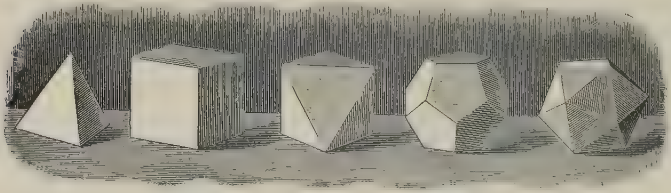
564. COROLLARY 1. *The volume of a frustum of a triangular pyramid may be expressed as* $\frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$.

For we may factor by $\frac{1}{3}a$.

565. COROLLARY 2. *The volume of a frustum of any pyramid is equal to the sum of the volumes of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*

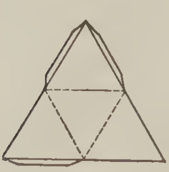
Extend the faces of the frustum F, forming a pyramid P. From a triangular pyramid P' of equivalent base b and equal altitude, cut off a frustum F' of the same altitude a as F. Then $P = P'$ and $F = F'$. But F and F' have equivalent bases, and $F' = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$. Hence $F = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$.

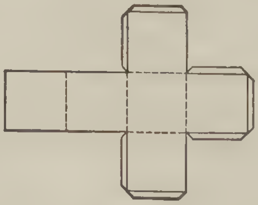
566. Polyhedrons classified as to Faces. A polyhedron of four faces is called a *tetrahedron;* one of six faces, a *hexahedron;* one of eight faces, an *octahedron;* one of twelve faces, a *dodecahedron;* one of twenty faces, an *icosahedron.*

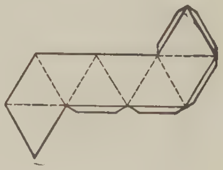

Tetrahedron Hexahedron Octahedron Dodecahedron Icosahedron

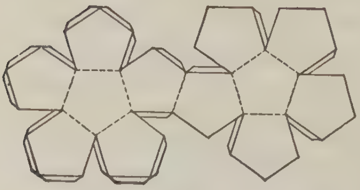
567. Regular Polyhedron. A polyhedron whose faces are congruent regular polygons, and whose polyhedral angles are equal, is called a *regular polyhedron.*

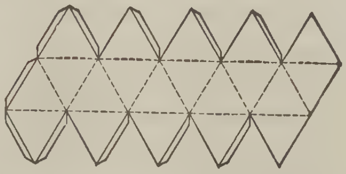
It is proved on page 351 that it is possible to have only five regular polyhedrons. They may be constructed from paper as follows:











Draw on stiff paper the diagrams given above. Cut through the full lines and paste strips of paper on the edges as shown. Fold on the dotted lines, and keep the edges in contact by the pasted strips of paper.

PROPOSITION XIX. PROBLEM

568. *To determine the number of regular convex polyhedrons possible.*

A convex polyhedral angle must have at least three faces, and the sum of its face angles must be less than 360° (§ 495).

1. Since each angle of an equilateral triangle is 60°, convex polyhedral angles may be formed by combining three, four, or five equilateral triangles. The sum of six such angles is 360°, and therefore is greater than the sum of the face angles of a convex polyhedral angle. Hence three regular convex polyhedrons are possible with equilateral triangles for faces.

2. Since each angle of a square is 90°, a convex polyhedral angle may be formed by combining three squares. The sum of four such angles is 360°, and therefore is greater than the sum of the face angles of a convex polyhedral angle. Hence one regular convex polyhedron is possible with squares.

3. Since each angle of a regular pentagon is 108° (§ 145), a convex polyhedral angle may be formed by combining three regular pentagons. The sum of four such angles is 432°, and therefore is greater than the sum of the face angles of a convex polyhedral angle. Hence one regular convex polyhedron is possible with regular pentagons.

4. The sum of three angles of a regular hexagon is 360°, of a regular heptagon is greater than 360°, and so on.

Hence only five regular convex polyhedrons are possible.

The regular polyhedrons are the regular *tetrahedron*, the regular *hexahedron*, or cube, the regular *octahedron*, the regular *dodecahedron*, and the regular *icosahedron*. Q.E.F.

It adds greatly to a clear understanding of the five regular polyhedrons if they are constructed from paper as suggested in § 567.

Since these solids were extensively studied by the pupils of Plato, the great Greek philosopher, they are often called the Platonic Bodies.

EXERCISE 92

Find the volumes of frustums of pyramids, the altitudes and the bases of the frustums being given, as follows:

1. $a = 3$ in., $b = 8$ sq. in., $b' = 2$ sq. in.
2. $a = 4\frac{1}{2}$ in., $b = 8\frac{1}{3}$ sq. in., $b' = 3$ sq. in.
3. $a = 3.2$ in., $b = 2$ sq. in., $b' = 0.18$ sq. in.
4. $a = 2$ ft. 6 in., $b = 10$ sq. ft., $b' = 2$ sq. ft. 72 sq. in.
5. $a = 3$ ft. 7 in., $b = 24$ sq. ft. 72 sq. in., $b' = 2$ sq. ft.

6. A pyramid 2 in. high, with a base whose area is 8 sq. in., is cut by a plane parallel to the base 1 in. from the vertex. Find the volume of the frustum.

7. A pyramid 3 in. high, with a base whose area is 81 sq. in., is cut by a plane parallel to the base 2 in. from the base. Find the volume of the frustum.

8. The lower base of a frustum of a pyramid is a square 4 in. on a side. The side of the upper base is half that of the lower base, and the altitude of the frustum is the same as the side of the upper base. Find the volume of the frustum.

9. The lower base of a frustum of a pyramid is a square 3 in. on a side. The area of the upper base is half that of the lower base, and the altitude of the frustum is 2 in. Find to two decimal places the volume of the frustum.

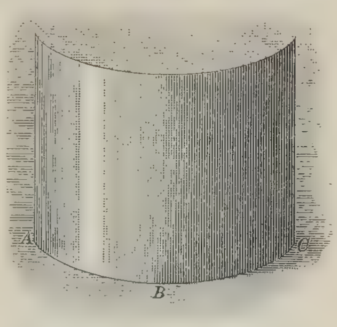
10. A pyramid has six edges, each 1 in. long. Find to two decimal places the volume of the pyramid.

11. A regular tetrahedron has a volume $2\sqrt{2}$ cu. in. Find to two decimal places the length of an edge.

12. The base of a regular pyramid is a square l ft. on a side. The slant height is s ft. Find the area of the entire surface.

13. Consider the formula $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$, of § 564, when $b' = 0$. Discuss the meaning of the result. Also discuss the case in which $b = b'$.

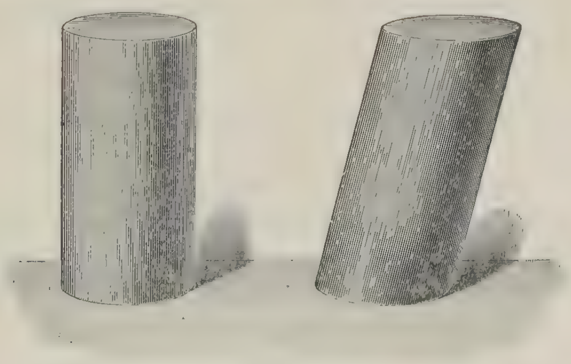
569. Cylindric Surface. A surface generated by a straight line which is constantly parallel to a fixed straight line, and touches a fixed curve not in the plane of the straight line, is called a *cylindric surface*, or a cylindrical surface.



The moving line is called the *generatrix* and the fixed curve the *directrix*. In the figure *ABC* is the directrix.

570. Element. The generatrix in any position is called an *element* of the cylindric surface.

571. Cylinder. A solid bounded by a cylindric surface and two parallel plane surfaces is called a *cylinder*.



It follows, therefore, *that all the elements of a cylinder are equal.*

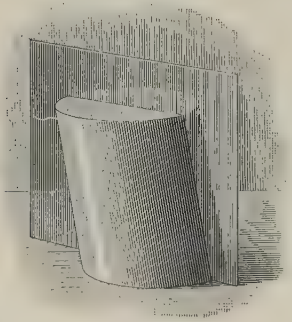
The terms *bases*, *lateral surface*, and *altitude* are used as with prisms.

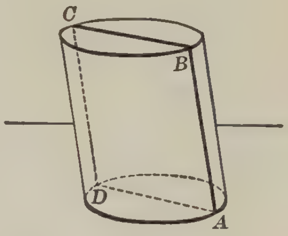
572. Right and Oblique Cylinders. A cylinder whose elements are perpendicular to its bases is called a *right cylinder;* otherwise a cylinder is called an *oblique cylinder*.

573. Section of a Cylinder. A figure formed by the intersection of a plane and a cylinder is called a *section of the cylinder*.

PROPOSITION XX. THEOREM

574. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*





Given a cylinder *AC*, and a section *ABCD* made by a plane passing through the element *AB*.

To prove that *ABCD is a parallelogram.*

Proof. Through D draw a line in the plane $ABCD \parallel$ to AB.

This line is an element of the cylindric surface. § 570

Since this line is in both the plane and the cylindric surface, it must be their intersection and must coincide with DC.

Hence DC coincides with a straight line parallel to AB.

Therefore DC is a straight line $\parallel$ to AB.

Also AD is a straight line $\parallel$ to BC. § 453

$\therefore$ $ABCD$ is a parallelogram, by § 118. Q.E.D.

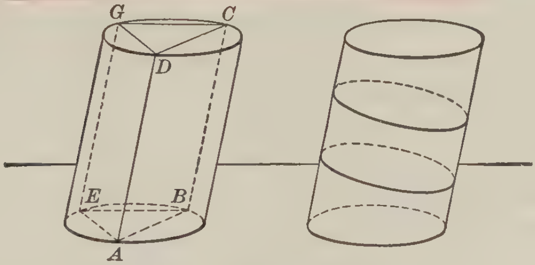
575. COROLLARY. *Every section of a right cylinder made by a plane passing through an element is a rectangle.*

576. Circular Cylinder. A cylinder whose bases are circles is called a *circular cylinder.*

A right circular cylinder, being generated by the revolution of a rectangle about one side as an axis, is also called a *cylinder of revolution.*

Proposition XXI. Theorem

577. *The bases of a cylinder are congruent.*



Given the cylinder AC, with bases ABE and DCG.

To prove that ABE *is congruent to* DCG.

Proof. Let A, B, E be any three points in the perimeter of the lower base, and AD, BC, EG be elements of the surface.

Draw AB, AE, EB, DC, DG, GC.

Then AD, BC, EG are equal, § 571

and parallel. § 569

$\therefore AB = DC$, $AE = DG$, $EB = GC$. § 130

$\therefore \triangle ABE$ is congruent to $\triangle DCG$. § 80

Place the lower base on the upper base so that the $\triangle ABE$ shall fall on the $\triangle DCG$. Then A, B, E will fall on D, C, G.

Therefore all points in either perimeter will coincide with points in the other, and the bases are congruent, by § 66. Q. E. D.

578. Corollary 1. *Any two parallel sections of a cylinder, cutting all the elements, are congruent.*

579. Corollary 2. *Any section of a cylinder parallel to the base is congruent to the base.*

580. Corollary 3. *The straight line joining the centers of the bases of a circular cylinder passes through the centers of all sections of the cylinder parallel to the bases.*

581. Tangent Plane. A plane which contains an element of a cylinder, but does not cut the surface, is called a *tangent plane* to the cylinder.

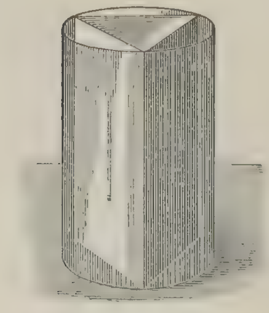
582. Construction of Tangent Planes. From a consideration of the nature of a tangent plane and of the construction of a cylindric surface it is evident that:

A plane passing through a tangent to the base of a circular cylinder and the element drawn through the point of contact is tangent to the cylinder.

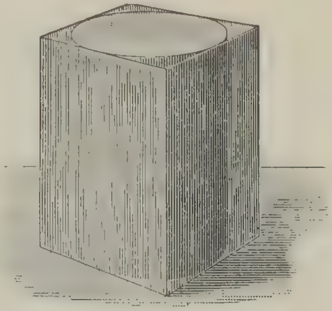
If a plane is tangent to a circular cylinder, its intersection with the plane of the base is tangent to the base.

583. Inscribed Prism. A prism whose lateral edges are elements of a cylinder and whose bases are inscribed in the bases of the cylinder is called an *inscribed prism.*

In this case the cylinder is said to be *circumscribed* about the prism.



Inscribed Prism



Circumscribed Prism

584. Circumscribed Prism. A prism whose lateral faces are tangent to the lateral surface of a cylinder and whose bases are circumscribed about the bases of the cylinder is called a *circumscribed prism.*

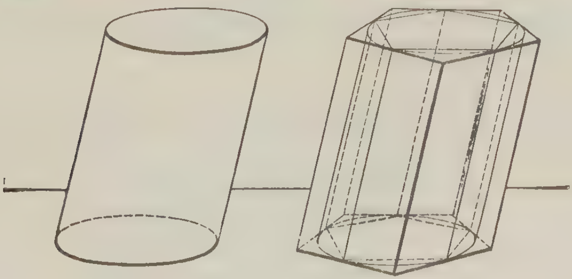
In this case the cylinder is said to be *inscribed* in the prism.

585. Right Section. A section of a cylinder made by a plane that cuts all the elements and is perpendicular to them is called a *right section* of the cylinder.

586. Cylinder as a Limit. From the work already done in connection with limits, and from the nature of the inscribed and circumscribed prisms, the following properties of the cylinder may now be assumed without further proof than that given below:

If a prism whose base is a regular polygon is inscribed in or circumscribed about a circular cylinder, and if the number of sides of the prism is indefinitely increased,

1. *The volume of the cylinder is the limit of the volume of the prism.*
2. *The lateral area of the cylinder is the limit of the lateral area of the prism.*
3. *The perimeter of a right section of the cylinder is the limit of the perimeter of a right section of the prism.*

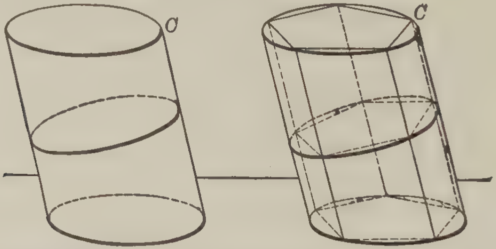


For as we increase the number of sides of the base of the inscribed or circumscribed prism whose base is a regular polygon, the perimeter of the base approaches the circle as its limit (§ 381).

This brings the lateral surface of each prism nearer and nearer the lateral surface of the cylinder. It also brings the volume of each prism nearer and nearer the volume of the cylinder. In the same way it brings the right section of each prism nearer and nearer the right section of the cylinder.

PROPOSITION XXII. THEOREM

587. *The lateral area of a circular cylinder is equal to the product of an element by the perimeter of a right section of the cylinder.*



Given a circular cylinder C, l being the lateral area, p the perimeter of a right section, and e an element.

To prove that $l = ep$.

Proof. Suppose a prism with base a regular polygon to be inscribed in C, l' being its lateral area and p' being the perimeter of its right section.

Then $l' = ep'$. § 512

If the number of lateral faces of the prism is indefinitely increased,

l' approaches l as a limit,

and p' approaches p as a limit, § 586

and consequently ep' approaches ep as a limit.

$\therefore l = ep$, by § 207. Q.E.D.

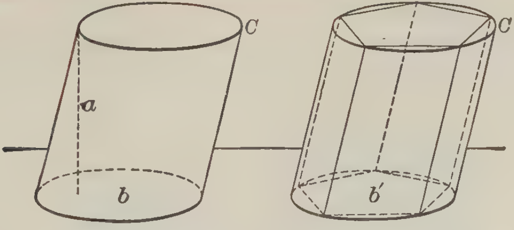
588. COROLLARY. *The lateral area of a cylinder of revolution is equal to the product of the altitude by the circumference of the base.*

In the case of a right circular cylinder of altitude a, lateral area l, total area t, and radius of base r, we have

$$l = 2\pi ra, \text{ and } t = 2\pi ra + 2\pi r^2 = 2\pi r(a + r).$$

PROPOSITION XXIII. THEOREM

589. *The volume of a circular cylinder is equal to the product of its base by its altitude.*



Given a circular cylinder C, b being the base, v the volume, and a the altitude.

To prove that $v = ba.$

Proof. Suppose a prism with base a regular polygon to be inscribed in C, b' being its base and v' being its volume.

Then $v' = b'a.$ § 539

If the number of lateral faces of the prism is indefinitely increased,

v' approaches v as a limit, § 586

b' approaches b as a limit, § 381

and consequently $b'a$ approaches ba as a limit.

But $v' = b'a$, whatever the number of sides. § 539

$\therefore v = ba$, by § 207. Q.E.D.

590. COROLLARY. *The volume of a cylinder of revolution with radius r and altitude a is $\pi r^2 a$.*

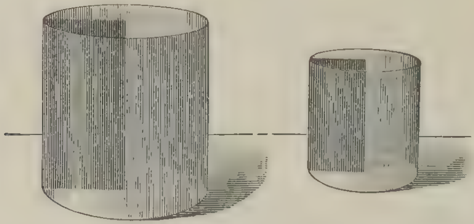
What is the area of the base? By what is this to be multiplied?

591. Similar Cylinders. Cylinders generated by the revolution of similar rectangles about corresponding sides are called *similar cylinders of revolution.*

§§ 591 and 592 may be omitted without destroying the sequence.

PROPOSITION XXIV. THEOREM

592. *The lateral areas, or the total areas, of similar cylinders of revolution are to each other as the squares of their altitudes or as the squares of their radii; and their volumes are to each other as the cubes of their altitudes or as the cubes of their radii.*



Given two similar cylinders of revolution, l and l' denoting their lateral areas, t and t' their total areas, v and v' their volumes, a and a' their altitudes, and r and r' their radii.

To prove that $l : l' = t : t' = a^2 : a'^2 = r^2 : r'^2$,

and that $v : v' = a^3 : a'^3 = r^3 : r'^3$.

Proof. Since the generating rectangles are similar, § 591

$$\therefore \frac{a}{a'} = \frac{r}{r'} = \frac{a+r}{a'+r'}. \qquad \text{§ 269}$$

Also we have by this proportion and § 588,

$$\frac{l}{l'} = \frac{2\pi ra}{2\pi r'a'} = \frac{ra}{r'a'} = \frac{r^2}{r'^2} = \frac{a^2}{a'^2}.$$

But $t = 2\pi ra + 2\pi r^2$ (§ 588), and $v = \pi r^2 a$. § 590

$$\therefore \frac{t}{t'} = \frac{2\pi r(a+r)}{2\pi r'(a'+r')} = \frac{r(a+r)}{r'(a'+r')} = \frac{r^2}{r'^2} = \frac{a^2}{a'^2},$$

and $$\frac{v}{v'} = \frac{\pi r^2 a}{\pi r'^2 a'} = \frac{r^2}{r'^2} \times \frac{a}{a'} = \frac{r^3}{r'^3} = \frac{a^3}{a'^3}.$$ Q.E.D.

EXERCISE 93

1. The diameter of a well is 6 ft. and the water is 7 ft. deep. How many gallons of water are there in the well, reckoning $7\frac{1}{2}$ gal. to the cubic foot?

2. When a body is placed under water in a right circular cylinder 60 centimeters in diameter, the level of the water rises 40 centimeters. Find the volume of the body.

3. How many cubic yards of earth must be removed in constructing a tunnel 100 yd. long, the section being a semicircle with a radius of 18 ft.?

4. How many square feet of sheet iron are required to make a pipe 18 in. in diameter and 40 ft. long?

5. Find the radius of a cylindric pail 14 in. high that will hold exactly 2 cu. ft.

6. The height of a cylindric vessel that will hold 20 liters is equal to the diameter. Find the altitude and the radius.

7. If the total surface of a right circular cylinder is t and the radius of the base is r, find the altitude a.

8. If the lateral surface of a right circular cylinder is l and the volume is v, find the radius r and the altitude a.

9. If the circumference of the base of a right circular cylinder is c and the altitude is a, find the volume v.

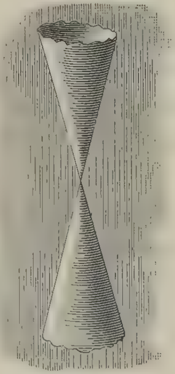
10. If the circumference of the base of a right circular cylinder is c and the total surface is t, find the volume v.

11. If the volume of a right circular cylinder is v and the altitude is a, find the total surface t.

12. If v is the volume of a right circular cylinder in which the altitude equals the diameter, find the altitude a and the total surface t.

13. From the formula $t = 2\pi r(a + r)$ (§ 588) find the value of r. (Omit unless quadratics have been studied.)

593. Conic Surface. A surface generated by a straight line which constantly touches a fixed plane curve and passes through a fixed point not in the plane of the curve is called a *conic surface* or a conical surface.



The moving line is called the *generatrix*, the fixed curve the *directrix*, and the fixed point the *vertex*.

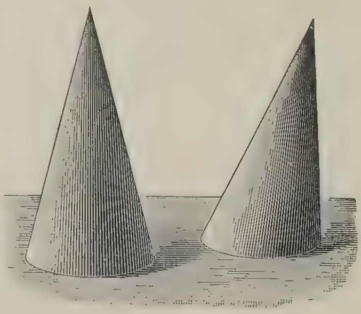
Hold a pencil by the point and let the other end swing around a circle, and the pencil will generate a conic surface.

We may also swing a blackboard pointer about any point near the middle, so that either end shall touch any fixed plane curve, and thus generate a conic surface. Such a surface is represented in the annexed figure.

594. Element. The generatrix in any position is called an *element* of the conic surface.

If the generatrix is of indefinite length, the surface consists of two portions, one above and the other below the vertex, which are called the *upper nappe* and *lower nappe* respectively. The two nappes are shown in the above figure.

595. Cone. A solid bounded by a conic surface and a plane cutting all the elements is called a *cone*.



The conic surface is called the *lateral surface* of the cone, and the plane surface is called the *base* of the cone.

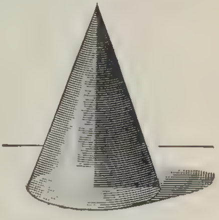
The vertex of the conic surface is called the *vertex* of the cone, and the elements of the conic surface are called the *elements* of the cone.

The perpendicular distance from the vertex to the plane of the base is called the *altitude* of the cone.

596. Circular Cone. A cone whose base is a circle is called a *circular cone.*

The straight line joining the vertex of a circular cone and the center of the base is called the *axis* of the cone.

597. Right and Oblique Cones. A circular cone whose axis is perpendicular to the base is called a *right cone;* otherwise a circular cone is called an *oblique cone.*



598. Cone of Revolution. Since a right circular cone may be generated by the revolution of a right triangle about one of the sides of the right angle, it is called a *cone of revolution.*

In this case the hypotenuse corresponds to an element of the surface and is called the *slant height.*

599. Conic Section. A section formed by the intersection of a plane and the conic surface of a cone of revolution is called a *conic section.*

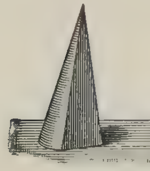


FIG. 1

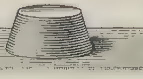


FIG. 2

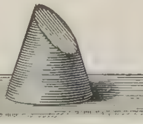


FIG. 3

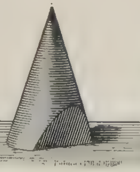


FIG. 4

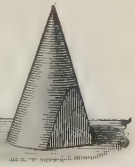


FIG. 5

In Fig. 1 the conic section is two intersecting straight lines, and this is discussed in § 600. This is true for all kinds of cones.

In Fig. 2 the conic section is a circle, and this is discussed in § 601.

In Fig. 3 the conic section is called an *ellipse*, the form a circle seems to take when looked at obliquely. The orbit of a planet is an ellipse.

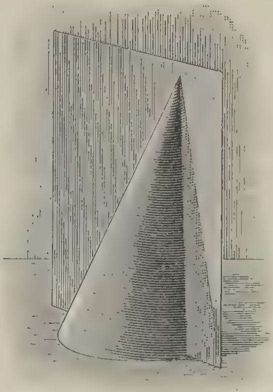
In Fig. 4 the conic section is a *parabola*, the path of a projectile (in a vacuum). Here the cutting plane is parallel to an element.

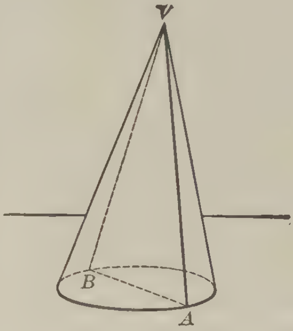
In Fig. 5 the conic section is an *hyperbola.*

The general study of conic sections is not a part of elementary geometry, but the names of the sections may profitably be known.

PROPOSITION XXV. THEOREM

600. *Every section of a cone made by a plane passing through its vertex is a triangle.*





Given a cone, with AVB a section made by a plane passing through the vertex V.

To prove that AVB *is a triangle.*

Proof. AB is a straight line. § 429

Draw the straight lines VA and VB.

The lines VA and VB are both elements of the surface of the given cone. § 594

These lines lie in the cutting plane, since their extremities are in the plane. § 422

Hence VA and VB are the intersections of the conic surface with the cutting plane.

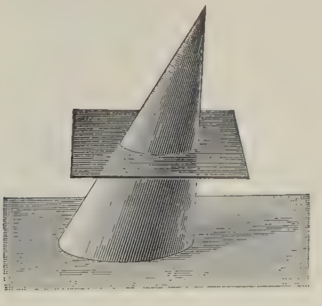
But VA and VB are straight lines. Const.

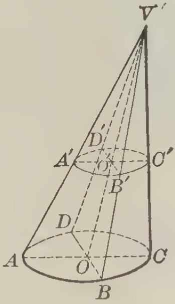
Therefore the intersections of the conic surface and the plane are straight lines.

Therefore the section AVB is a triangle, by § 28. Q.E.D.

PROPOSITION XXVI. THEOREM

601. *In a circular cone a section made by a plane parallel to the base is a circle.*





Given the circular cone *V-ABCD*, with the section *A′B′C′D′* parallel to the base.

To prove that $A'B'C'D'$ *is a circle.*

Proof. Let O be the center of the base, and let O' be the point in which the axis VO pierces the plane of the conic section.

Through VO and any elements VA, VB, pass planes cutting the base in the radii OA, OB, and cutting the section $A'B'C'D'$ in the straight lines $O'A'$, $O'B'$.

Then $O'A'$ and $O'B'$ are $\parallel$ respectively to OA and OB. § 453

Therefore the $\triangle$ AOV and OBV are similar respectively to the $\triangle$ $A'O'V$ and $O'B'V$. § 285

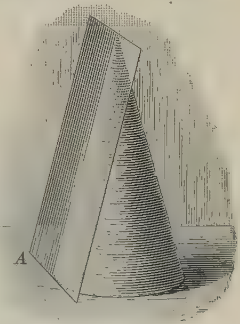
$$\therefore \frac{OA}{O'A'} = \frac{VO}{VO'} = \frac{OB}{O'B'}. \qquad \text{§ 282}$$

But $OA = OB$. § 162

$\therefore O'A' = O'B'$ (§ 263), and $A'B'C'D'$ is a circle, by § 159. Q.E.D.

602. COROLLARY. *The axis of a circular cone passes through the center of every section which is parallel to the base.*

603. Tangent Plane. A plane which contains an element of a cone, but does not cut the surface, is called a *tangent plane* to the cone.



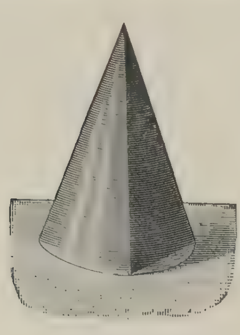
604. Construction of Tangent Planes. It is evident that:

A plane passing through a tangent to the base of a circular cone and the element drawn through the point of contact is tangent to the cone.

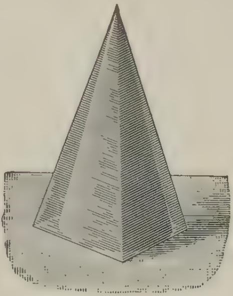
If a plane is tangent to a circular cone its intersection with the plane of the base is tangent to the base.

605. Inscribed Pyramid. A pyramid whose lateral edges are elements of a cone and whose base is inscribed in the base of the cone is called an *inscribed pyramid.*

In this case the cone is said to be *circumscribed* about the pyramid.



Inscribed Pyramid

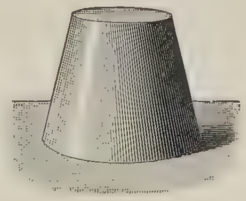


Circumscribed Pyramid

606. Circumscribed Pyramid. A pyramid whose lateral faces are tangent to the lateral surface of a cone and whose base is circumscribed about the base of the cone is called a *circumscribed pyramid.*

In this case the cone is said to be *inscribed* in the pyramid.

607. Frustum of a Cone. The portion of a cone included between the base and a section parallel to the base is called a *frustum of a cone.*

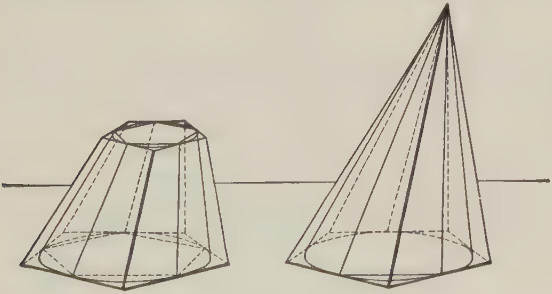


The base of the cone and the parallel section are together called the *bases* of the frustum.

The terms *altitude* and *lateral area* of a frustum of a cone, and *slant height* of a frustum of a right circular cone, are used in substantially the same manner as with the frustum of a pyramid (§§ 550, 551, 552).

608. Cones and Frustums as Limits. The following properties, similar to those of § 586, are assumed without proof:

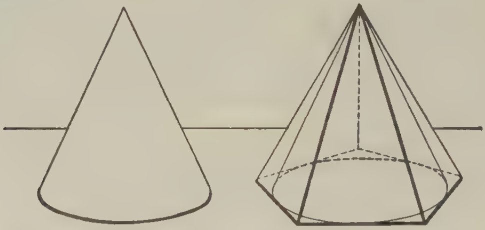
If a pyramid whose base is a regular polygon is inscribed in or circumscribed about a circular cone, and if the number of sides of the base of the pyramid is indefinitely increased, the volume of the cone is the limit of the volume of the pyramid, and the lateral area of the cone is the limit of the lateral area of the pyramid.



The volume of a frustum of a cone is the limit of the volumes of the frustums of the inscribed and circumscribed pyramids, if the number of lateral faces is indefinitely increased, and the lateral area of the frustum of a cone is the limit of the lateral areas of the frustums of the inscribed and circumscribed pyramids, the bases being regular polygons.

PROPOSITION XXVII. THEOREM

609. *The lateral area of a cone of revolution is equal to half the product of the slant height by the circumference of the base.*



Given a cone of lateral area l, circumference of base c, and slant height s.

To prove that $l = \frac{1}{2}sc$.

Proof. Suppose a regular pyramid to be circumscribed about the cone, the perimeter of its base being p and its lateral area l'.

Then $l' = \frac{1}{2}sp$. § 553

If the number of the lateral faces of the circumscribed pyramid is indefinitely increased,

l' approaches l as a limit, § 608

p approaches c as a limit, § 381

and consequently $\frac{1}{2}sp$ approaches $\frac{1}{2}sc$ as a limit.

But $l' = \frac{1}{2}sp$, whatever the number of sides. § 553

$\therefore l = \frac{1}{2}sc$, by § 207. Q.E.D.

610. COROLLARY. *If l denotes the lateral area, t the total area, s the slant height, and r the radius of the base of a cone of revolution, then*

$$l = \tfrac{1}{2}(2\pi r \times s) = \pi rs;$$

$$t = \pi rs + \pi r^2 = \pi r(s + r).$$

EXERCISE 94

Find the lateral areas of cones of revolution, given the slant heights and the circumferences of the bases respectively as follows:

1. $2\frac{7}{8}$ in., $5\frac{3}{8}$ in.
2. $4\frac{3}{8}$ in., $8\frac{1}{4}$ in.
3. $6\frac{5}{16}$ in., $10\frac{1}{2}$ in.
4. 3.7 in., 5.8 in.
5. 5.3 in., 9.7 in.
6. 6.5 in., 11.6 in.
7. 2 ft. 6 in., 4 ft. 8 in.
8. 3 ft. 7 in., 8 ft. 6 in.
9. 5 ft. 8 in., 12 ft. 4 in.

Find the lateral areas of cones of revolution, given the slant heights and the radii of the bases respectively as follows:

10. $3\frac{3}{4}$ in., $2\frac{1}{2}$ in.
11. $2\frac{1}{2}$ in., $1\frac{3}{4}$ in.
12. $4\frac{7}{8}$ in., $3\frac{1}{4}$ in.
13. 6.4 in., 4.8 in.
14. 7.2 in., 5.3 in.
15. 8.9 in., 5.6 in.
16. 2 ft. 3 in., 8 in.
17. 4 ft. 6 in., 2 ft.
18. 6 ft. 9 in., 3 ft. 2 in.

Find the total areas of cones of revolution, given the slant heights and the radii of the bases respectively as follows:

19. 3 in., 2 in.
20. 5 in., 3 in.
21. 7 in., 4 in.
22. 9 in., 5 in.
23. 6 ft., 4 ft.
24. 12 ft., 5 ft.

25. Deduce a formula for finding the lateral area of a cone of revolution in terms of the radius of the base and the altitude.

26. Deduce a formula for finding the slant height in terms of the lateral area and the circumference of the base.

27. Deduce a formula for finding the slant height in terms of the lateral area and the radius of the base.

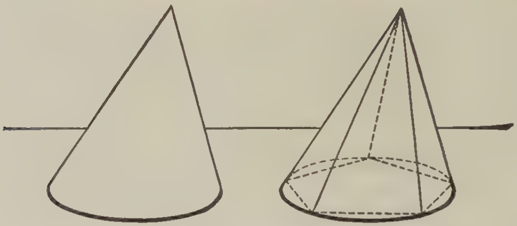
28. Deduce a formula for finding the radius of the base in terms of the lateral area and the slant height.

29. Deduce a formula for finding the slant height in terms of the total area and the radius of the base.

30. Deduce a formula for finding the circumference of the base in terms of the lateral area and the slant height.

PROPOSITION XXVIII. THEOREM

611. *The volume of a circular cone is equal to one third the product of its base by its altitude.*



Given a circular cone of volume v, base b, and altitude a.

To prove that $v = \frac{1}{3}ba$.

Proof. Suppose a pyramid with base a regular polygon to be inscribed in the cone, b' being its base and v' its volume.

Then $v' = \frac{1}{3}b'a$. § 561

If the number of lateral faces of the pyramid is indefinitely increased,

v' approaches v as a limit, § 608

b' approaches b as a limit, § 381

and consequently $b'a$ approaches ba as a limit.

$\therefore v = \frac{1}{3}ba$, by § 207. Q.E.D.

612. COROLLARY. *In a circular cone of radius r and altitude a, $v = \frac{1}{3}\pi r^2 a$.*

For the area of the base is πr^2 (§ 389).

613. Similar Cones. Cones generated by the revolution of similar right triangles about corresponding sides are called *similar cones* of revolution.

In case § 614 is omitted this definition may also be omitted.

EXERCISE 95

Find the volumes of circular cones, given the altitudes and the areas of the bases respectively as follows:

1. 4 in., 8 sq. in.

2. $3\frac{1}{4}$ in., $9\frac{3}{8}$ sq. in.

3. $5\frac{3}{8}$ in., $10\frac{1}{2}$ sq. in.

4. 6.3 in., 3.8 sq. in.

5. 7.8 in., 6.9 sq. in.

6. 9.3 in., 16.8 sq. in.

Find the volumes of circular cones, given the altitudes and the radii of the bases respectively as follows:

7. 4 in., 3 in.

8. 6 in., 4 in

9. 8 in., 5 in

10. 9.8 in., 4.3 in.

11. 10.5 in., 6.2 in.

12. 14.9 in., 9.6 in.

13. How many cubic feet in a conical tent 10 ft. in diameter and 7 ft. high?

14. How many cubic feet in a conical pile of earth 15 ft. in diameter and 8 ft. high?

15. Deduce a formula for finding the altitude of a circular cone in terms of the volume and the area of the base.

16. Deduce a formula for finding the area of the base of a circular cone in terms of the volume and the altitude.

17. Deduce a formula for finding the altitude of a circular cone in terms of the volume and the radius of the base.

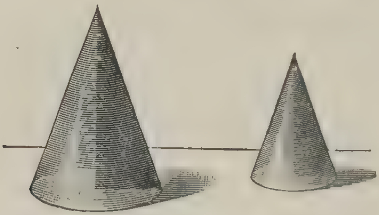
18. Deduce a formula for finding the radius of the base of a circular cone in terms of the volume and the altitude.

19. Deduce a formula for finding the volume of a cone of revolution in terms of the slant height and the radius of the base.

20. Deduce formulas for finding the slant height and the altitude of a cone of revolution in terms of the volume and the radius of the base.

PROPOSITION XXIX. THEOREM

614. *The lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their altitudes, as the squares of their radii, or as the squares of their slant heights; and their volumes are to each other as the cubes of their altitudes, as the cubes of their radii, or as the cubes of their slant heights.*



Given two similar cones of revolution, with lateral areas l and l', total areas t and t', volumes v and v', altitudes a and a', radii r and r', and slant heights s and s' respectively.

To prove that $l : l' = t : t' = a^2 : a'^2 = r^2 : r'^2 = s^2 : s'^2$,

and that $v : v' = a^3 : a'^3 = r^3 : r'^3 = s^3 : s'^3$.

Proof. $$\frac{a}{a'} = \frac{r}{r'} = \frac{s}{s'} = \frac{s+r}{s'+r'}. \qquad \S\S\ 282, 269$$

$$\frac{l}{l'} = \frac{\pi rs}{\pi r's'} = \frac{r}{r'} \times \frac{s}{s'} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2} = \frac{a^2}{a'^2}. \qquad \S\ 610$$

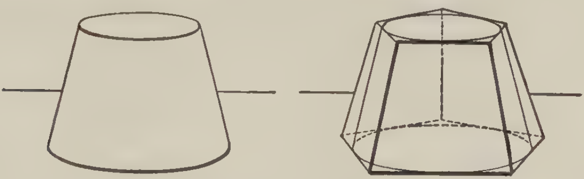
$$\frac{t}{t'} = \frac{\pi r(s+r)}{\pi r'(s'+r')} = \frac{r}{r'} \times \frac{s+r}{s'+r'} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2} = \frac{a^2}{a'^2}. \qquad \S\ 610$$

$$\frac{v}{v'} = \frac{\frac{1}{3}\pi r^2 a}{\frac{1}{3}\pi r'^2 a'} = \frac{r^2}{r'^2} \times \frac{a}{a'} = \frac{r^3}{r'^3} = \frac{a^3}{a'^3} = \frac{s^3}{s'^3}, \text{ by } \S\ 612. \qquad \text{Q.E.D.}$$

§§ 613 and 614, like §§ 591 and 592, are occasionally demanded in college entrance examinations. They are not needed for any exercises and they may be omitted without destroying the sequence.

PROPOSITION XXX. THEOREM

615. *The lateral area of a frustum of a cone of revolution is equal to half the sum of the circumferences of its bases multiplied by the slant height.*



Given a frustum of a cone of revolution, with lateral area l, circumferences of bases c and c', and slant height s.

To prove that $l = \frac{1}{2}(c + c')s$.

Proof. Suppose a frustum of a regular pyramid circumscribed about the frustum of the cone, as a pyramid is circumscribed about a cone.

Let the lateral area of the circumscribed frustum be l', and let p and p' be the perimeters of the bases corresponding to c and c' respectively. The slant height is s, the same as that of the frustum of the cone.

Then $l' = \frac{1}{2}(p + p')s$. § 554

If the number of lateral faces of the circumscribed frustum is indefinitely increased, what limits do l' and $p + p'$ approach?

Therefore what limit does $\frac{1}{2}(p + p')s$ approach?

What conclusion may be drawn, as in § 587?

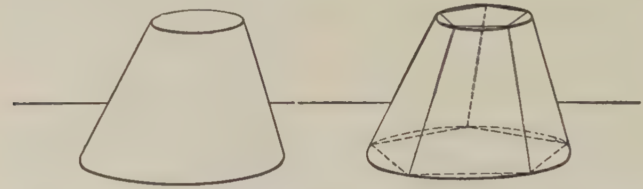
Complete the proof.

616. COROLLARY. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equidistant from its bases multiplied by its slant height.*

How can it be proved that $\frac{1}{2}(c + c')$ equals the circumference of this section? How are the radii related?

PROPOSITION XXXI. THEOREM

617. *A frustum of a circular cone is equivalent to the sum of three cones whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*



Given a frustum of a circular cone, with volume v, bases b and b', and altitude a.

To prove that $v = \frac{1}{3} a(b + b' + \sqrt{bb'})$.

Proof. Suppose a frustum of a pyramid with base a regular polygon to be inscribed in the frustum of the cone, as a pyramid is inscribed in a cone.

Let v' be the volume, and let x and x' be the bases corresponding to b and b' respectively. The altitude is a, the same as that of the frustum of the cone.

Then $v' = \frac{1}{3} a(x + x' + \sqrt{xx'})$. § 565

If the number of lateral faces of the inscribed frustum is indefinitely increased, what limits do v', x, x', and xx' approach?

Therefore what limit does $\frac{1}{3} a(x + x' + \sqrt{xx'})$ approach?

What conclusion may be drawn?

Complete the proof.

618. COROLLARY. *In a frustum of a cone of revolution, r and r' being the radii of the bases,* $v = \frac{1}{3} \pi a(r^2 + r'^2 + rr')$.

For $b = \pi r^2$, $b' = \pi r'^2$. $\therefore \sqrt{bb'} = \sqrt{\pi r^2 \times \pi r'^2} = \pi rr'$.

EXERCISE 96

Find the lateral areas of frustums of cones, given the circumferences of the bases and the slant heights respectively as follows:

1. $c = 4$ in., $c' = 3$ in., $s = 0.5$ in.

2. $c = 6$ in., $c' = 5$ in., $s = 1.4$ in.

3. $c = 7\frac{1}{2}$ in., $c' = 5\frac{3}{4}$ in., $s = 2\frac{1}{8}$ in.

4. $c = 23$ in., $c' = 18$ in., $s = 16$ in.

Find to two decimal places the volumes of frustums of cones, given the altitudes and the areas of the bases respectively as follows:

5. $a = 3$ in., $b = 4\frac{1}{2}$ sq. in., $b' = 2$ sq. in.

6. $a = 4$ in., $b = 8\frac{1}{3}$ sq. in., $b' = 3$ sq. in.

7. $a = 5\frac{1}{2}$ in., $b = 16$ sq. in., $b' = 9$ sq. in.

8. $a = 6$ in., $b = 17$ sq. in., $b' = 11$ sq. in.

Find to two decimal places the volumes of frustums of cones of revolution, given the altitudes and the radii of the bases respectively as follows:

9. $a = 4$ in., $r = 3$ in., $r' = 2$ in.

10. $a = 5$ in., $r = 3\frac{1}{2}$ in., $r' = 2\frac{1}{4}$ in.

11. $a = 6$ in., $r = 3.7$ in., $r' = 3.1$ in.

12. $a = 7\frac{1}{2}$ in., $r = 4\frac{3}{4}$ in., $r' = 3\frac{1}{8}$ in.

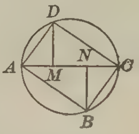
13. Deduce a formula for finding the altitude of a frustum of a circular cone in terms of the volume and the areas of the bases.

14. Deduce a formula for finding the altitude of a frustum of a cone of revolution in terms of the volume and the radii of the bases.

EXERCISE 97

INDUSTRIAL PROBLEMS

1. There is a rule for calculating the strongest beam that can be cut from a cylindric log, as follows:



Erect perpendiculars MD and NB on opposite sides of a diameter AC, at the trisection points M and N, meeting the circle in D and B. Then $ABCD$ is a section of the beam.

Calculate the dimensions, the log being 16 in. in diameter.

2. A cylindric funnel for a steamboat is 4 ft. 3 in. in diameter. It is built up of four plates in girth, and the lap of each joint is $1\frac{3}{8}$ in. Find one dimension of each plate.

3. A tubular boiler has 124 tubes each $3\frac{7}{8}$ in. in diameter and 18 ft. long. Required the total tube surface. Answer to the nearest square foot.

4. A room in a factory is heated by steam pipes. There are 235 ft. of 2-inch pipe and 26 ft. 3 in. of 3-inch pipe, besides 2 ft. 8 in. of $4\frac{1}{4}$-inch feed pipe. Required the total heating surface. Answer to the nearest square foot.

5. A triangular plate of wrought iron $\frac{5}{8}$ in. thick is 2 ft. 7 in. on each side. If the weight of a plate 1 ft. square and $\frac{1}{8}$ in. thick is 5 lb., find to the nearest pound the weight of the given triangular plate.

6. The water surface of an upright cylindric boiler is 2 ft. 8 in. below the top of the boiler, and is 12.57 sq. ft. in area. What is the volume of the steam space?

7. A cylinder 16 in. in diameter is required to hold 50 gal. of water. What must be its height, to the nearest tenth of an inch, allowing 231 cu. in. to the gallon?

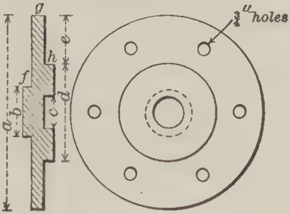
8. How many square feet of tin are required to make a funnel, if the diameters of the top and bottom are 30 in. and 15 in. respectively, and the height is 25 in.?

9. Find to two decimal places the weight of a steel plate 4 ft. by 3 ft. 2 in. by $1\frac{3}{8}$ in., allowing 490 lb. per cubic foot.

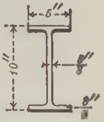
10. A steel plate for a steamship is 5 ft. long, 3 ft. 6 in. wide, and $\frac{1}{2}$ in. thick. A porthole 10 in. in diameter is cut through the plate. Required the weight of the finished plate, allowing 0.29 lb. per cubic inch. Answer to two decimal places.

11. A cast-iron base for a column is in the form of a frustum of a pyramid, the lower base being a square 2 ft. on a side, and the upper base having a fourth of the area of the lower base. The altitude of the frustum is 9 in. Required the weight to the nearest pound, allowing 460 lb. per cubic foot.

12. A cylinder head for a steam engine has the shape shown in the figure, where the dimensions in inches are: $a = 12$, $b = 3$, $c = 2$, $d = 6$, $e = 3$, $f = \frac{1}{2}$, $g = \frac{7}{8}$, and $h = \frac{3}{4}$. There are six $\frac{3}{4}$-inch holes for bolts. Compute the weight of the plate, allowing 41 lb. for the weight of a steel plate 1 ft. square and 1 in. thick. Answer to the nearest tenth of a pound.



13. A steel beam 10 in. by 5 in., in the form here shown, is 18 ft. long. The thickness of the beam is $\frac{5}{8}$ in. and the average thickness of the flanges is $\frac{3}{8}$ in. Find the weight of the beam to the nearest pound, allowing 0.29 lb. per cubic inch.



14. A hollow steel shaft 12 ft. long is 18 in. in exterior diameter and 8 in. in interior diameter. Find the weight to the nearest pound, allowing 0.29 lb. per cubic inch.

15. Find the expense, at 70 cents a square foot, of polishing the curved surface of a marble column in the shape of the frustum of a right circular cone whose slant height is 12 ft. and the radii of whose bases are 3 ft. 6 in. and 2 ft. 4 in. respectively.

EXERCISE 98

Miscellaneous Problems

1. The slant height of the frustum of a regular pyramid is 25 ft., and the sides of its square bases are 54 ft. and 24 ft. respectively. Find the volume.

2. If the bases of the frustum of a pyramid are regular hexagons whose sides are 1 ft. and 2 ft. respectively, and the volume of the frustum is 12 cu. ft., find the altitude.

3. From a right circular cone whose slant height is 30 ft., and the circumference of whose base is 10 ft., there is cut off by a plane parallel to the base a cone whose slant height is 6 ft. Find the lateral area and the volume of the frustum.

4. Find the difference between the volume of the frustum of a pyramid whose altitude is 9 ft. and whose bases are squares, 8 ft. and 6 ft. respectively on a side, and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them.

5. A Dutch stone windmill in the shape of the frustum of a right cone is 12 meters high. The outer diameters at the bottom and the top are 16 meters and 12 meters, the inner diameters 12 meters and 10 meters. How many cubic meters of stone were required to build it?

6. The chimney of a factory has the shape of a frustum of a regular pyramid. Its height is 180 ft., and its upper and lower bases are squares whose sides are 10 ft. and 16 ft. respectively. The flue throughout is a square whose side is 7 ft. How many cubic feet of material does the chimney contain?

7. Two right triangles with bases 15 in. and 21 in., and with hypotenuses 25 in. and 35 in. respectively, revolve about their third sides. Find the ratio of the total areas of the solids generated and find their volumes.

EXERCISE 99

Equivalent Solids

1. A cube each edge of which is 12 in. is transformed into a right prism whose base is a rectangle 16 in. long and 12 in. wide. Find the height of the prism and the difference between its total area and the total area of the cube.

2. The dimensions of a rectangular parallelepiped are a, b, c. Find the height of an equivalent right circular cylinder, having a for the radius of its base; the height of an equivalent right circular cone having a for the radius of its base.

3. A regular pyramid 12 ft. high is transformed into a regular prism with an equivalent base. Find the height of the prism.

4. The diameter of a cylinder is 14 ft. and its height 8 ft. Find the height of an equivalent right prism, the base of which is a square with a side 4 ft. long.

5. If one edge of a cube is e, what is the height h of an equivalent right circular cylinder whose radius is r?

6. The heights of two equivalent right circular cylinders are in the ratio $4:9$. If the diameter of the first is 6 ft., what is the diameter of the second?

7. A right circular cylinder 6 ft. in diameter is equivalent to a right circular cone 7 ft. in diameter. If the height of the cone is 8 ft., what is the height of the cylinder?

8. The frustum of a regular pyramid 6 ft. high has for bases squares 5 ft. and 8 ft. on a side. Find the height of an equivalent regular pyramid whose base is a square 12 ft. on a side.

9. The frustum of a cone of revolution is 5 ft. high and the diameters of its bases are 2 ft. and 3 ft. respectively. Find the height of an equivalent right circular cylinder whose base is equal in area to the section of the frustum made by a plane parallel to the bases and equidistant from them.

EXERCISE 100

REVIEW QUESTIONS

1. Define polyhedron. Is a cylinder a polyhedron?

2. Define prism, and classify prisms according to their bases.

3. How is the lateral area of a prism computed? Is the method the same for right as for oblique prisms?

4. Define parallelepiped; rectangular parallelepiped; cube. Is a rectangular parallelepiped always a cube? Is a cube always a rectangular parallelepiped?

5. Distinguish between equivalent and congruent solids. Are two cubes with the same altitudes always equivalent? always congruent? Is this true for parallelepipeds?

6. What are the conditions of congruence of two prisms? of two right prisms? of two cubes?

7. The opposite angles of a parallelogram are equal. What is a corresponding proposition concerning parallelepipeds?

8. How do you find the volume of a parallelepiped? What is the corresponding proposition in plane geometry?

9. How do you find the volume of a prism? of a cylinder? of a pyramid? of a cone?

10. Define pyramid. How many bases has a pyramid? Is there any kind of a pyramid in which more than one face may be taken as the base?

11. How do you find the lateral area of a pyramid? of a right cone? of a frustum of a pyramid? of a frustum of a right cone?

12. How many regular convex polyhedrons are possible? What are their names?

13. Given the radius of the base and the altitude of a cone of revolution, how do you find the volume? the lateral area? the total area?

流
天球
地球

BOOK VIII

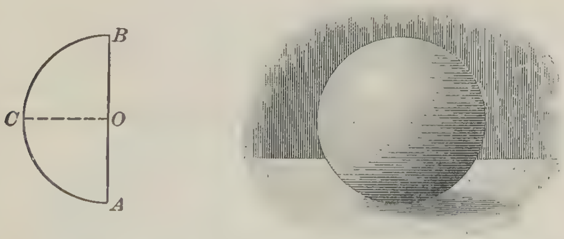
THE SPHERE

619. Sphere. A solid bounded by a surface all points of which are equidistant from a point within is called a *sphere.*

The point within, from which all points on the surface are equally distant, is called the *center*. The surface is called the *spherical surface*, and sometimes the *sphere*. Half of a sphere is called a *hemisphere*. The terms *radius* and *diameter* are used as in the case of a circle.

620. Generation of a Spherical Surface. By the definition of sphere it appears that a spherical surface may be generated by the revolution of a semicircle about its diameter as an axis.

Thus, if the semicircle ACB revolves about AB, a spherical surface is generated. It is therefore assumed that *a sphere may be described with any given point as a center and any given line as a radius.*



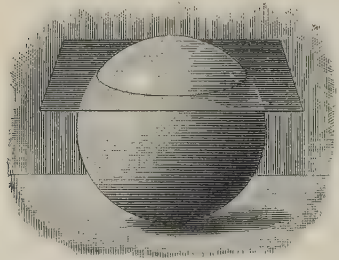
621. Equality of Radii and Diameters. It follows that:

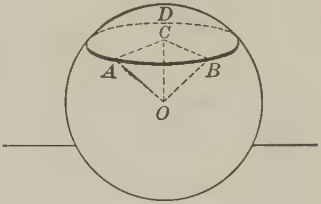
All radii of the same sphere are equal, and all diameters of the same sphere are equal.

Equal spheres have equal radii, and spheres having equal radii are equal.

PROPOSITION I. THEOREM

622. *Every intersection of a spherical surface by a plane is a circle.*





Given a sphere with center *O*, and *ABD* any section of its surface made by a plane.

To prove that the section ABD is a circle.

Proof. Draw the radii OA, OB, to any two points A, B, in the section, and draw $OC \perp$ to the plane of the section.

Then in $\triangle s$ OCA and OCB, $\angle s$ OCA and OCB are rt. $\angle s$, § 430

OC is common, and $OA = OB$. § 621

$\therefore \triangle OCA$ is congruent to $\triangle OCB$. § 89

$\therefore CA = CB$. § 67

$\therefore$ any points A and B, and hence all points, in the section are equidistant from C, and ABD is a ⊙, by § 159. Q.E.D.

623. COROLLARY 1. *The line joining the center of a sphere and the center of a circle of the sphere is perpendicular to the plane of the circle.*

624. COROLLARY 2. *Circles of a sphere made by planes equidistant from the center are equal; and of two circles made by planes not equidistant from the center the one made by the plane nearer the center is the greater.*

625. Great Circle. The intersection of a spherical surface by a plane passing through the center is called a *great circle* of the sphere.

626. Small Circle. The intersection of a spherical surface by a plane which does not pass through the center is called a *small circle* of the sphere.

627. Poles of a Circle. If a diameter of a sphere is perpendicular to the plane of a circle of the sphere, the extremities are called the *poles* of the circle.

628. COROLLARY 1. *Parallel circles have the same poles.*

629. COROLLARY 2. *All great circles of a sphere are equal.*

630. COROLLARY 3. *Every great circle bisects the spherical surface.*

631. COROLLARY 4. *Two great circles bisect each other.*

The intersection of the planes passes through what point?

632. COROLLARY 5. *If the planes of two great circles are perpendicular, each circle passes through the poles of the other.*

Draw the figure and state the reason.

633. COROLLARY 6. *Through two given points on the surface of a sphere an arc of a great circle may always be drawn.*

Do these two points, together with the center of the sphere, generally determine a plane? Consider the special case in which the two points are ends of a diameter.

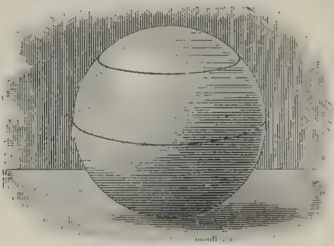
634. COROLLARY 7. *Through three given points on the surface of a sphere one circle and only one can be drawn.*

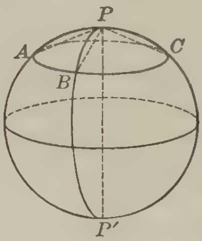
How many points determine a plane?

635. Spherical Distance. The length of the smaller arc of the great circle joining two points on the surface of a sphere is called the *spherical distance* between the points, or, where no confusion is likely to arise, simply the *distance.*

PROPOSITION II. THEOREM

636. *The spherical distances of all points on a circle of a sphere from either pole of the circle are equal.*





Given P, P', the poles of the circle ABC, and A, B, C, any points on the circle.

To prove that the great-circle arcs PA, PB, PC are equal.

Proof. The straight lines PA, PB, PC are equal. § 439

Therefore the arcs PA, PB, PC are equal, by § 172. Q. E. D.

In like manner, the great-circle arcs $P'A$, $P'B$, $P'C$ may be proved equal.

637. Polar Distance. The spherical distance from the nearer pole of a circle to any point on the circle is called the *polar distance of the circle.*

The spherical distance of a great circle from either of its poles may be taken as the polar distance of the circle.

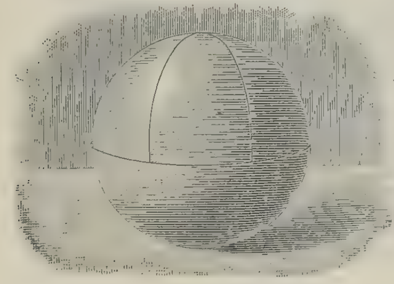
638. Quadrant. One fourth of a great circle is called a *quadrant.*

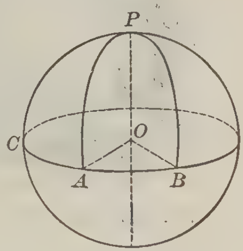
639. COROLLARY 1. *The polar distance of a great circle is a quadrant.*

640. COROLLARY 2. *The straight lines joining points on a circle to either pole of the circle are equal.*

PROPOSITION III. THEOREM

641. *A point on a sphere, which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is a pole of the great circle passing through these points.*





Given a point P on a sphere, PA and PB quadrants, and ABC the great circle passing through A and B.

To prove that P *is the pole of* $\odot ABC$.

Proof. What kind of angles are the $\angle s\ AOP$ and BOP?

How is PO related to the plane of $\odot ABC$?

Does this prove that P is the pole of $\odot ABC$?

642. Describing Circles on a Sphere. This proposition proves that we may describe a great circle on a sphere of a given radius so that it shall pass through two given points.

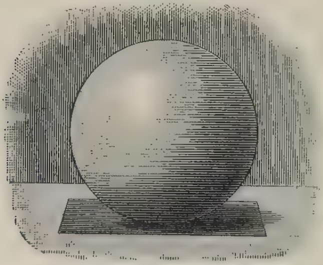
Open the compasses the length of chord $PA = \sqrt{r^2 + r^2} = r\sqrt{2}$.

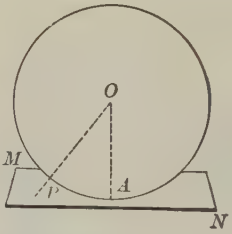
643. Tangent Lines and Planes. A line or plane that has one point and only one point in common with a sphere, however far produced, is said to be *tangent* to the sphere, and the sphere to be *tangent* to the line or plane.

644. Tangent Spheres. Two spheres whose surfaces have one point and only one point in common are said to be *tangent*.

PROPOSITION IV. THEOREM

645. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*





Given the plane *MN* perpendicular to the radius *OA* at *A*.

To prove that MN is tangent to the sphere.

Proof. Let P be any point except A in MN.

Then which is longer, OP or OA, and why?

Therefore, is P inside, on, or outside the sphere, and why?

What does this tell us concerning all points, except A, on MN?

How, then, do we know that MN is tangent to the sphere?

646. COROLLARY. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

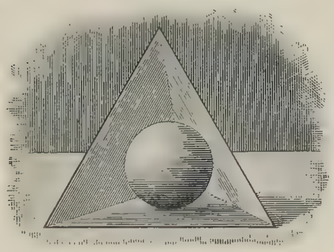
What are the proposition and corollary of plane geometry corresponding to §§ 645 and 646? Do they suggest the proof of this corollary?

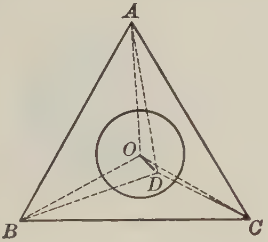
647. Inscribed Sphere. If a sphere is tangent to all the faces of a polyhedron, it is said to be *inscribed* in the polyhedron, and the polyhedron to be circumscribed about the sphere.

648. Circumscribed Sphere. If all the vertices of a polyhedron lie on a spherical surface, the sphere is said to be *circumscribed* about the polyhedron, and the polyhedron to be inscribed in the sphere.

Proposition V. Theorem

649. *A sphere may be inscribed in any given tetrahedron.*





Given the tetrahedron ***ABCD.***

To prove that a sphere may be inscribed in ABCD.

Proof. Bisect the dihedral $\angle$s at the edges *AB*, *BC*, and *CA* by the planes *OAB*, *OBC*, and *OCA* respectively.

Every point in the plane *OAB* is equidistant from the faces *ABC* and *ABD*. § 479

For a like reason every point in the plane *OBC* is equidistant from the faces *ABC* and *DBC*; and every point in the plane *OCA* is equidistant from the faces *ABC* and *ADC*.

Therefore the point *O*, the common intersection of these three planes, is equidistant from the four faces of the tetrahedron and is the center of the sphere inscribed in the tetrahedron, by § 647. Q.E.D.

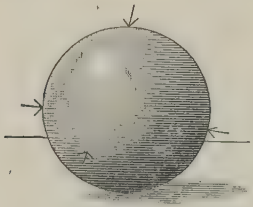
Discussion. What is the corresponding proposition in plane geometry? Is the line of proof similar?

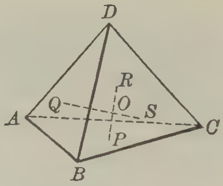
It is shown in plane geometry that the three lines which bisect the three angles of a triangle meet in a point. What is the corresponding proposition with reference to planes in a tetrahedron? Is it substantially proved in this proposition?

It is proved in plane geometry that a circle may be inscribed in what kind of a polygon? What corresponding proposition may be inferred in solid geometry?

Proposition VI. Theorem

650. *A sphere may be circumscribed about any given tetrahedron.*





Given the tetrahedron $ABCD$.

To prove that a sphere may be circumscribed about $ABCD$.

Proof. Let P, Q respectively be the centers of the circles circumscribed about the faces ABC, ABD.

Let PR be $\perp$ to the face ABC, and $QS \perp$ to the face ABD.

Then PR is the locus of a point equidistant from A, B, C, and QS is the locus of a point equidistant from A, B, D. § 442

Therefore PR and QS lie in the same plane, the plane $\perp$ to AB at its mid-point. § 443

If QS were $\parallel$ to PR, it would be $\perp$ to the face ABC. § 445

But this is impossible, for QS is $\perp$ to the face ABD which intersects the face ABC. Given

Since PR and QS cannot be $\parallel$, and since they lie in the same plane, they must therefore meet at some point O.

$\therefore$ O is equidistant from A, B, C, and D,

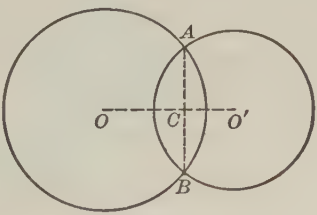
and is the center of the required sphere, by § 648. Q.E.D.

651. Corollary. *Through four points not in the same plane one spherical surface and only one can be passed.*

The center of any sphere whose surface passes through the four points must be in the perpendiculars mentioned in the proof, and since there is only one point of intersection, there can be only one sphere.

PROPOSITION VII. THEOREM

652. *The intersection of two spherical surfaces is a circle whose plane is perpendicular to the line which joins the centers of the spheres and whose center is in that line.*



Given two intersecting spherical surfaces, with centers O and O'.

To prove that the spherical surfaces intersect in a circle whose plane is perpendicular to OO', and whose center is in OO'.

Proof. Let the two great circles formed by any plane through O and O' intersect in A and B.

Then OO' is a ⊥ bisector of AB. § 195

If this plane revolves about OO', the circles generate the spherical surfaces, and A describes their line of intersection.

But during the revolution AC remains constant in length and perpendicular to OO'.

Therefore A generates a circle with center C, whose plane is perpendicular to OO', by § 432. Q.E.D.

653. Spherical Angle. The opening between two great-circle arcs that intersect is called a *spherical angle.* A spherical angle is considered equal to the plane angle formed by the tangents to the arcs at their point of intersection.

Draw a figure illustrating this definition.

In elementary geometry we do not consider angles formed by arcs of small circles.

EXERCISE 101

1. The four perpendiculars erected at the centers of the circles circumscribed about the faces of a tetrahedron meet in the same point.

2. The six planes perpendicular to the edges of a tetrahedron at their mid-points intersect in the same point.

3. The six planes which bisect the six dihedral angles of a tetrahedron intersect in the same point.

4. Circles on the same sphere having equal polar distances are equal.

5. Equal circles on the same sphere have equal polar distances.

6. Find the locus of a point in a plane at a given distance from a given point. Also of a point in a three-dimensional space.

7. A line tangent to a great circle of a sphere lies in the plane tangent to the sphere at the point of contact.

8. Any line in a tangent plane drawn through the point of contact is tangent to the sphere at that point.

9. One plane and only one plane can be passed through a given point on a given sphere tangent to the sphere.

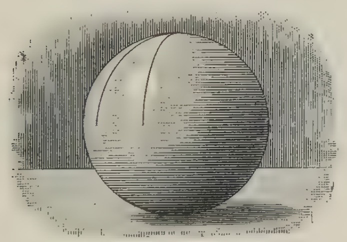
10. Find a point in a plane equidistant from two intersecting lines in the plane, and at a given distance from a given point not in the plane. Discuss the solution.

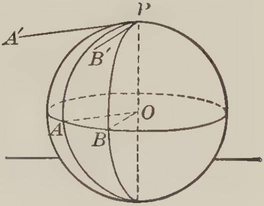
11. How many points determine a straight line? a circle? a spherical surface? Prove that two spherical surfaces coincide if they have this number of points in common.

12. If two planes which intersect in the line AB touch a sphere at the points C and D respectively, the line CD is perpendicular to AB in the sense mentioned in the discussion under § 450, — that a plane can be passed through CD perpendicular to AB.

PROPOSITION VIII. THEOREM

654. *A spherical angle is measured by the arc of the great circle described from its vertex as a pole and included between its sides, produced if necessary.*





Given PA and PB, arcs of great circles intersecting at P; PA' and PB', the tangents to these arcs at P; AB, the arc of the great circle described from P as a pole and included between PA and PB.

To prove that the spherical $\angle APB$ *is measured by arc* AB.

Proof. In the plane POB, PB' is $\perp$ to PO, § 185

and OB is $\perp$ to PO. § 213

$\therefore PB'$ is $\parallel$ to OB. § 95

Similarly PA' is $\parallel$ to OA.

$\therefore \angle A'PB' = \angle AOB$. § 461

But $\angle AOB$ is measured by arc AB. § 213

$\therefore \angle A'PB'$ is measured by arc AB.

$\therefore \angle APB$ is measured by arc AB, by § 653. Q.E.D.

655. COROLLARY 1. *A spherical angle has the same measure as the dihedral angle formed by the planes of the two circles.*

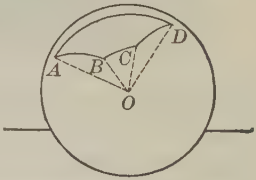
656. COROLLARY 2. *All arcs of great circles drawn through the pole of a given great circle are perpendicular to the given circle.*

657. Spherical Polygon. A portion of a spherical surface bounded by three or more arcs of great circles is called a *spherical polygon.*

The bounding arcs are called the *sides* of the polygon, the angles between the sides are called the *angles* of the polygon, and the points of intersection of the sides are called the *vertices* of the polygon.

658. Relation of Polygons to Polyhedral Angles. The planes of the sides of a spherical polygon form a polyhedral angle whose vertex is the center of the sphere, whose face angles are measured by the sides of the polygon, and whose dihedral angles have the same numerical measure as the angles of the polygon.

Thus the planes of the sides of the polygon $ABCD$ form the polyhedral angle $O\text{-}ABCD$. The face angles BOA, COB, and so on, are measured by the sides AB, BC, and so on, of the polygon. The dihedral angle whose edge is OA has the same measure as the spherical angle BAD, and so on.



Hence from any property of polyhedral angles we may infer an analogous property of spherical polygons; and conversely.

659. Convex Spherical Polygon. If a polyhedral angle at the center of a sphere is convex (§ 491), the corresponding spherical polygon is said to be *convex.*

Every spherical polygon is assumed to be convex unless the contrary is stated.

660. Diagonal. An arc of a great circle joining two nonconsecutive vertices of a spherical polygon is called a *diagonal.*

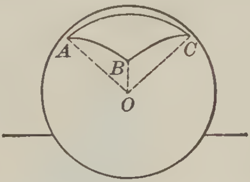
661. Spherical Triangle. A spherical polygon of three sides is called a *spherical triangle.*

A spherical triangle may be *right*, *obtuse*, or *acute*. It may also be *equilateral*, *isosceles*, or *scalene*.

662. Congruent Spherical Polygons. If two spherical polygons can be applied, one to the other, so as to coincide, they are said to be *congruent.*

PROPOSITION IX. THEOREM

663. *Each side of a spherical triangle is less than the sum of the other two sides.*



Given a spherical triangle *ABC*, *CA* being the longest side.

To prove that $CA < AB + BC$.

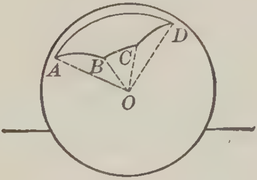
Proof. In the corresponding trihedral angle $O\text{-}ABC$,

$\angle COA$ is less than $\angle BOA + \angle COB$. § 494

$\therefore CA < AB + BC$, by § 658. Q. E. D.

PROPOSITION X. THEOREM

664. *The sum of the sides of a spherical polygon is less than 360°.*



Given a spherical polygon *ABCD*.

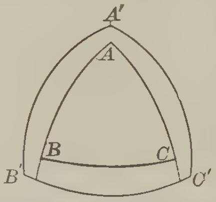
To prove that $AB + BC + CD + DA < 360°$.

Proof. In the corresponding polyhedral angle $O\text{-}ABCD$,

$\angle BOA + \angle COB + \angle DOC + \angle DOA < 360°$. § 495

$\therefore AB + BC + CD + DA < 360°$, by § 658. Q. E. D.

665. Polar Triangle. If from the vertices of a spherical triangle as poles arcs of great circles are described, another spherical triangle is formed which is called the *polar triangle* of the first.



Thus, if A is the pole of the arc of the great circle $B'C'$, B of $C'A'$, C of $A'B'$, $A'B'C'$ is the polar triangle of ABC.

If, with A, B, C as poles, entire great circles are described, these circles divide the surface of the sphere into *eight* spherical triangles.

Of these eight triangles, that one is the polar of ABC whose vertex A', corresponding to A, lies on the same side of BC as the vertex A; and similarly for the other vertices.

EXERCISE 102

1. To bisect a given great-circle arc.

What must be done to the angle at the center?

2. If two great-circle arcs intersect, the vertical angles are equal.

3. To describe an arc of a great circle through a given point and perpendicular to a given arc of a great circle.

4. Every point lying on a great circle which bisects a given arc of another great circle at right angles is equidistant (§ 635) from the extremities of the given arc.

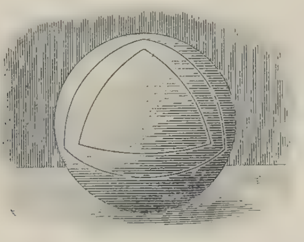
5. Two sides of a spherical triangle are respectively 82° 47′ and 67° 39′. What is known concerning the number of degrees in the third side?

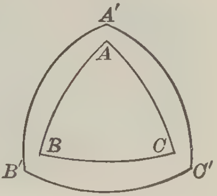
6. Three sides of a spherical quadrilateral are respectively 86° 29′, 73° 47′, and 69° 54′. What is known concerning the number of degrees in the fourth side?

7. Draw a picture of a sphere, and of an equilateral spherical triangle on the sphere, each side being 90°. Then draw a picture of the polar triangle.

PROPOSITION XI. THEOREM

666. *If one spherical triangle is the polar triangle of another, then reciprocally the second is the polar triangle of the first.*





Given the triangle ABC and its polar triangle $A'B'C'$.

To prove that ABC is the polar triangle of $A'B'C'$.

Proof. Since A is the pole of $B'C'$,
and C is the pole of $A'B'$, § 665
$\therefore$ B' is at a quadrant's distance from A and C. § 639
$\therefore$ B' is the pole of arc AC. § 641
Similarly A' is the pole of BC,
and C' is the pole of AB.
$\therefore$ ABC is the polar triangle of $A'B'C'$, by § 665. Q.E.D.

Discussion. Is it necessary that one of the triangles should be wholly within the other? Draw the figures approximately, without using instruments, starting with $\triangle ABC$ having $AB = 100°$, $AC = 100°$, $BC = 30°$.

Also draw the figures having $AB = 120°$, $AC = 80°$, $BC = 40°$.

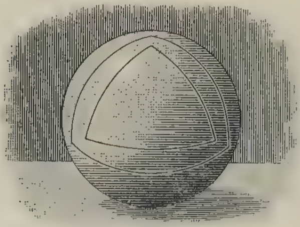
Also draw the figures suggested in Ex. 7, on page 394, where $AB = BC = CA = 90°$. Consider the proposition with these figures.

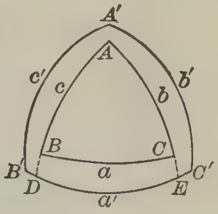
The proposition may also be considered by starting with $\triangle ABC$ as the polar triangle of $\triangle A'B'C'$, and proving that $\triangle A'B'C'$ is the polar triangle of $\triangle ABC$.

It is desirable in the study of spherical triangles to have a spherical blackboard. Where this is not available, any wooden ball will serve the purpose.

PROPOSITION XII. THEOREM

667. *In two polar triangles each angle of the one is the supplement of the opposite side in the other.*





Given two polar triangles ABC and $A'B'C'$, the letter at the vertex of each angle denoting its value in degrees, and the small letter denoting the value of the opposite side in degrees.

To prove that $A + a' = 180°$, $B + b' = 180°$, $C + c' = 180°$;
$A' + a = 180°$, $B' + b = 180°$, $C' + c = 180°$.

Proof. Produce the arcs AB, AC until they meet $B'C'$ at the points D, E respectively.

Since B' is the pole of AE, $\therefore B'E = 90°$. § 639

And since C' is the pole of AD, $\therefore DC' = 90°$.

$\therefore B'E + DC' = 180°$. Ax. 1

That is, $B'D + DE + DC' = 180°$, Ax. 9

or $DE + B'C' = 180°$. Ax. 9

But DE is the measure of the $\angle A$, § 654

and $B'C' = a'$.

$\therefore A + a' = 180°$.

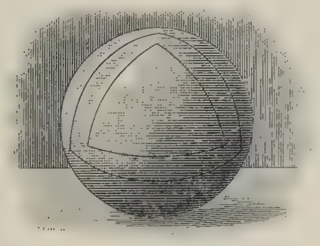
Similarly $B + b' = 180°$,

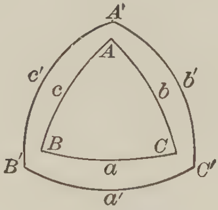
and $C + c' = 180°$.

In a similar way, starting with $\triangle A'B'C'$ and producing the sides of $\triangle ABC$, all the other relations are proved. Q.E.D.

PROPOSITION XIII. THEOREM

668. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*





Given a spherical triangle *ABC*, the letter at the vertex of each angle denoting its value in degrees, and the small letter denoting the value of the opposite side in degrees.

To prove that $A + B + C > 180°$ *and* $< 540°$.

Proof. Let $\triangle A'B'C'$ be the polar triangle of $\triangle ABC$.

Then $A + a' = 180°, B + b' = 180°, C + c' = 180°.$ § 667

$\therefore A + B + C + a' + b' + c' = 540°.$ Ax. 1

$\therefore A + B + C = 540° - (a' + b' + c').$ Ax. 2

Now $a' + b' + c' < 360°.$ § 664

$\therefore A + B + C = 540° -$ some value less than $360°.$

$\therefore A + B + C > 180°.$

Again $a' + b' + c'$ is greater than $0°.$

$\therefore A + B + C < 540°.$ Q.E.D.

669. COROLLARY. *A spherical triangle may have two, or even three, right angles; and a spherical triangle may have two, or even three, obtuse angles.*

670. Triangles classified as to Right Angles. A spherical triangle having two right angles is said to be *birectangular;* one having three right angles is said to be *trirectangular.*

The same terms may be applied to the corresponding trihedral angles.

EXERCISE 103

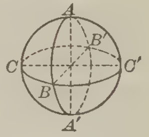
1. If two sides of a spherical triangle are quadrants, the third side measures the opposite angle.

2. In a birectangular spherical triangle the sides opposite the right angles are quadrants, and the side opposite the third angle measures that angle.

Since the $\angle$s are rt. $\angle$s, what two planes are $\perp$ to a third plane ? What two arcs must therefore (§ 632) pass through the pole of a third arc ? Then what two arcs are quadrants ? Then how is the third angle (§ 654) measured ?

3. Each side of a trirectangular spherical triangle is a quadrant.

4. Three planes passed through the center of a sphere, each perpendicular to the other two, divide the spherical surface into eight congruent trirectangular triangles.



Find the number of degrees in the sides of a spherical triangle, given the angles of its polar triangle as follows:

5. 82°, 77°, 69°.
6. $84\tfrac{1}{2}^\circ, 81\tfrac{3}{4}^\circ, 72\tfrac{1}{6}^\circ$.
7. 78° 30′, 89°, 102°.
8. 83° 40′, 48° 57′, 103° 43′.
9. 96° 37′ 40″, 82° 29′ 30″, 68° 47′.
10. 43° 29′ 37″, 98° 22′ 53″, 87° 36′ 39″.

Find the number of degrees in the angles of a spherical triangle, given the sides of its polar triangle as follows:

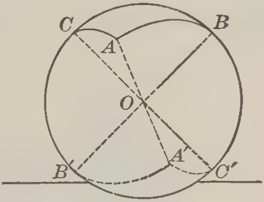
11. 68° 42′ 39″, 93° 48′ 7″, 89° 38′ 14″.

12. 78° 47′ 29″, 106° 36′ 42″, a quadrant.

13. A quadrant, half a quadrant, three fourths of a quadrant.

14. From the center of a sphere are drawn three radii, each perpendicular to the other two. Find the number of degrees in the sides and angles of the spherical triangle determined by their extremities.

671. Symmetric Spherical Triangles. If through the center O of a sphere three diameters AA', BB', CC' are drawn, and the points A, B, C are joined by arcs of great circles, and also the points A', B', C', the two spherical triangles ABC and $A'B'C'$ are called *symmetric spherical triangles.*

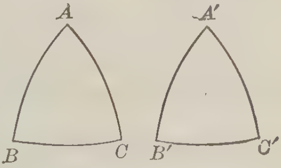


In the same way we may form two symmetric polygons of any number of sides. Having thus formed the symmetric polygons, we may place them in any position we choose upon the surface of the sphere.

672. Relation of Symmetric Triangles. Two symmetric triangles are mutually equilateral and mutually equiangular; yet in general they are not congruent, for they cannot be made to coincide by superposition. If in the above figure the triangle ABC is made to slide on the surface of the sphere until the vertex A falls on A', it is evident that the two triangles cannot be made to coincide for the reason that the corresponding parts of the triangles occur in *reverse order.*

To try to make two symmetric spherical polygons coincide is very much like trying to put the right-hand glove on the left hand. The relation of two symmetric spherical triangles may be illustrated by cutting them out of the peel of an orange or an apple.

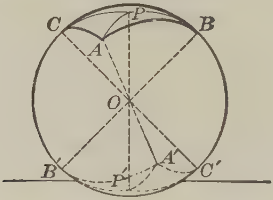
673. Symmetric Isosceles Triangles. If, however, we have two symmetric triangles ABC and $A'B'C'$ such that $AB = AC$, and $A'B' = A'C'$, that is, if the two symmetric triangles are *isosceles*, then because AB, AC, $A'B'$, $A'C'$ are all equal and the angles A and A' are equal, being originally formed by vertical dihedral angles (§ 671), the two triangles can be made to coincide. Therefore,



If two symmetric spherical triangles are isosceles, they are superposable and therefore are congruent.

PROPOSITION XIV. THEOREM

674. *Two symmetric spherical triangles are equivalent.*



Given two symmetric spherical triangles ABC, $A'B'C'$, having their corresponding vertices opposite each to each with respect to the center of the sphere.

To prove that the triangles ABC, $A'B'C'$ are equivalent.

Proof. Let P be the pole of a small circle passing through the points A, B, C, and let POP' be a diameter.

Draw the great-circle arcs PA, PB, PC, $P'A'$, $P'B'$, $P'C'$.

Then $PA = PB = PC$. § 636

Now $P'A' = PA$, $P'B' = PB$, $P'C' = PC$. § 672

$\therefore P'A' = P'B' = P'C'$. Ax. 8

$\therefore$ the two symmetric $\triangle$s PCA and $P'C'A'$ are isosceles.

$\therefore \triangle PCA$ is congruent to $\triangle P'C'A'$. § 673

Similarly $\triangle PAB$ is congruent to $\triangle P'A'B'$,

and $\triangle PBC$ is congruent to $\triangle P'B'C'$.

Now $\triangle ABC = \triangle PCA + \triangle PAB + \triangle PBC$,

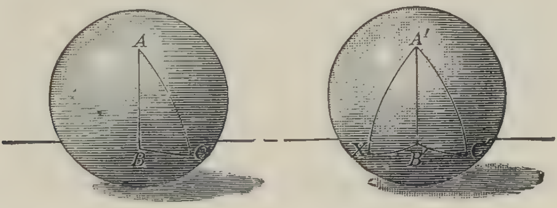
and $\triangle A'B'C' = \triangle P'C'A' + \triangle P'A'B' + \triangle P'B'C'$. Ax. 11

$\therefore \triangle ABC$ is equivalent to $\triangle A'B'C'$, by Ax. 9. Q.E.D.

Discussion. If the pole P should fall without the $\triangle ABC$, then P' would fall without $\triangle A'B'C'$, and each triangle would be equivalent to the sum of two symmetric isosceles triangles diminished by the third; so that the result would be the same as before.

PROPOSITION XV. THEOREM

675. *Two triangles on the same sphere or on equal spheres are either congruent or symmetric if two sides and the included angle of the one are respectively equal to the corresponding parts of the other.*



Given two spherical triangles ABC and $A'B'C'$, with $AB = A'B'$, $AC = A'C'$, and angle A = angle A', and similarly arranged; and given the triangle $A'B'X$ symmetric with respect to the triangle $A'B'C'$.

To prove that $\triangle ABC$ *is congruent to* $\triangle A'B'C'$, *and that* $\triangle ABC$ *is symmetric with respect to* $\triangle A'B'X$.

Proof. Superpose $\triangle ABC$ on $\triangle A'B'C'$, the proof being similar to that of the corresponding case in plane geometry. § 68

$\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$. § 662

Since $\triangle A'B'X$ is symmetric with respect to $\triangle A'B'C'$,

and $\triangle ABC$ is congruent to $\triangle A'B'C'$,

$$\therefore AC = A'X,\ AB = A'B',\ \angle A = \angle XA'B'.$$

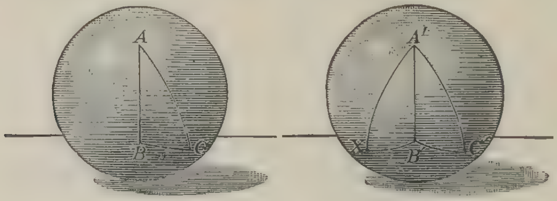
But $\triangle ABC$ is congruent to $\triangle A'B'C'$ and may be made to coincide with it.

$\therefore \triangle ABC$ is symmetric with respect to $\triangle A'B'X$. Q.E.D.

Discussion. In the case of plane triangles, if the corresponding parts are arranged in reverse order, we can still prove the triangles congruent. Why can we not do so in the case of spherical triangles?

Proposition XVI. Theorem

676. *Two triangles on the same sphere or on equal spheres are either congruent or symmetric if two angles and the included side of the one are respectively equal to the corresponding parts of the other.*



Given two spherical triangles ABC and $A'B'C'$, with angle $A =$ angle A', angle $C =$ angle C', and $AC = A'C'$, and similarly arranged; and given the triangle $A'B'X$ symmetric with respect to the triangle $A'B'C'$.

To prove that $\triangle ABC$ is congruent to $\triangle A'B'C'$, and that $\triangle ABC$ is symmetric with respect to $\triangle A'B'X$.

Proof. Superpose $\triangle ABC$ on $\triangle A'B'C'$, the proof being similar to that of the corresponding case in plane geometry. § 72

$\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$. § 662

Since $\triangle A'B'X$ is symmetric with respect to $\triangle A'B'C'$, and $\triangle ABC$ is congruent to $\triangle A'B'C'$,

$$\therefore \angle A = \angle XA'B', \ \angle C = \angle X, \text{ and } AC = A'X.$$

But $\triangle ABC$ is congruent to $\triangle A'B'C'$ and may be made to coincide with it.

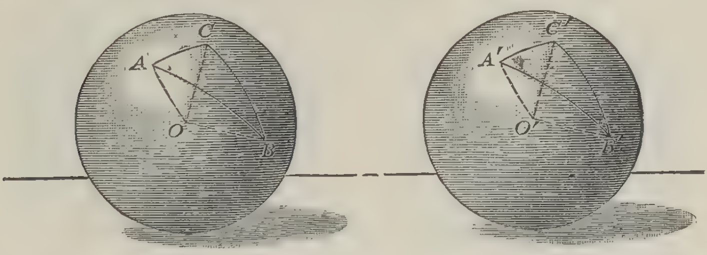
$\therefore \triangle ABC$ is symmetric with respect to $\triangle A'B'X$. Q.E.D.

Discussion. Under what circumstances are the two triangles both congruent and symmetric?

In plane geometry what is the case that corresponds to the one in which the spherical triangles are both congruent and symmetric?

PROPOSITION XVII. THEOREM

677. *Two mutually equilateral triangles on the same sphere or on equal spheres are mutually equiangular, and are either congruent or symmetric.*



Given two spherical triangles, ABC, $A'B'C'$, on equal spheres, such that $AB = A'B'$, $BC = B'C'$, $CA = C'A'$.

To prove that $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, and that $\triangle$s ABC and $A'B'C'$ are either congruent or symmetric.

Proof. Let O and O' be the centers of the spheres.

Pass a plane through each pair of vertices of each triangle and the center of its sphere.

Then in the trihedral angles at O and O' the face angles are equal each to its corresponding face angle. § 167

$\therefore$ the corresponding dihedral $\angle$s are respectively equal. § 499

$\therefore$ the $\angle$s of the spherical $\triangle$s are respectively equal. § 655

$\therefore$ the $\triangle$s are either congruent or symmetric, by § 676. Q.E.D.

Discussion. In the figures the parts are arranged in the same order, so that the triangles are congruent. They might be arranged as in the figures of § 676.

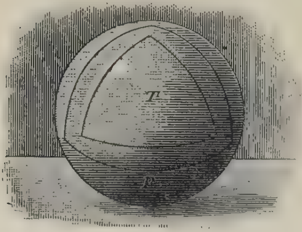
Discuss the proposition when the triangles are equilateral and each side is a quadrant.

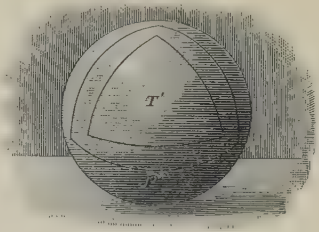
Discuss the proposition when two sides of each triangle are quadrants.

What is the corresponding proposition in plane geometry, and why does not the form of proof there given hold here?

Proposition XVIII. Theorem

678. *Two mutually equiangular triangles on the same sphere or on equal spheres are mutually equilateral, and are either congruent or symmetric.*





Given two mutually equiangular spherical triangles T and T' on equal spheres.

To prove that T and T' are mutually equilateral, and are either congruent or symmetric.

Proof. Let the $\triangle P$ be the polar triangle of $\triangle T$, and the $\triangle P'$ be the polar triangle of $\triangle T'$.

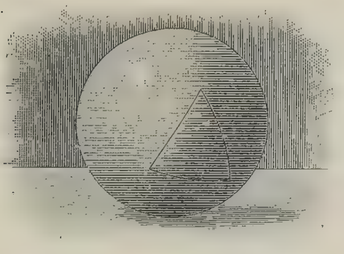
Since the $\triangle$ T and T' are mutually equiangular, Given
$\therefore$ the polar $\triangle$ P and P' are mutually equilateral. § 667
$\therefore$ the polar $\triangle$ P and P' are mutually equiangular. § 677
But the $\triangle$ T and T' are the polar $\triangle$ of $\triangle$ P and P'. § 666
$\therefore$ the $\triangle$ T and T' are mutually equilateral. § 667

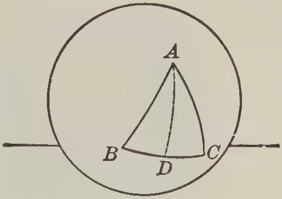
Therefore the $\triangle$ T and T' are either congruent or symmetric, by § 677. Q. E. D.

Discussion. The statement that mutually equiangular spherical triangles are mutually equilateral, and are either congruent or symmetric, is true only when they are on the same sphere or on equal spheres. When the spheres are unequal, the spherical triangles are unequal. In this case, however, their sides have the same arc measure, and therefore have the same ratio as the circumferences or as the radii of the spheres (§ 382).

PROPOSITION XIX. THEOREM

679. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*





Given the spherical triangle *ABC*, with *AB* equal to *AC*.

To prove that $\angle B = \angle C$.

Proof. Draw the arc AD of a great circle, from the vertex A to the mid-point of the base BC.

Then $\triangle$ ABD and ACD are mutually equilateral.

$\therefore$ $\triangle$ ABD and ACD are mutually equiangular. § 677

$\therefore \angle B = \angle C$. Q.E.D.

EXERCISE 104

1. The radius of a sphere is 4 in. From any point on the surface as a pole a circle is described upon the sphere with an opening of the compasses equal to 3 in. Find the area of this circle.

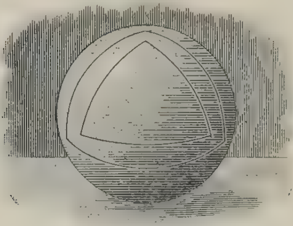
2. The edge of a regular tetrahedron is a. Find the radii r, r' of the inscribed and circumscribed spheres.

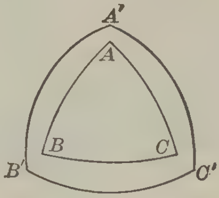
3. Find the diameter of the section of a sphere of diameter 10 in. made by a plane 3 in. from the center.

4. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the mid-point of the base bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetric triangles.

Proposition XX. Theorem

680. *If two angles of a spherical triangle are equal, the sides opposite these angles are equal and the triangle is isosceles.*





Given the spherical triangle ABC, with angle B equal to angle C.

To prove that $AC = AB$.

Proof. Let $\triangle A'B'C'$ be the polar triangle of $\triangle ABC$.

Since $\angle B = \angle C$, $\therefore A'C' = A'B'$. § 667

$\therefore \angle B' = \angle C'$. § 679

$\therefore AC = AB$, by § 667. Q.E.D.

EXERCISE 105

1. To bisect a given spherical angle.

2. To construct a spherical triangle, given two sides and the included angle.

3. To construct a spherical triangle, given two angles and the included side.

4. To construct a spherical triangle, given the three sides.

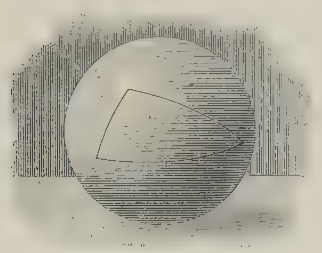
5. To construct a spherical triangle, given the three angles.

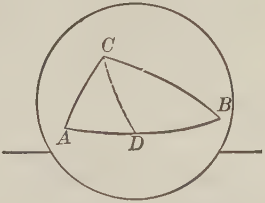
6. To pass a plane tangent to a given sphere at a given point on the surface of the sphere.

7. To pass a plane tangent to a given sphere through a given straight line without the sphere.

PROPOSITION XXI. THEOREM

681. *If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater; and if two sides are unequal, the angles opposite these sides are unequal, and the angle opposite the greater side is the greater.*





Given the triangle ABC, with angle C greater than angle B.

To prove that $AB > AC$.

Proof. Draw the arc CD of a great circle, making $\angle DCB$ equal to $\angle B$. Then $DB = DC$. § 680

Now $AD + DC > AC$. § 663

$\therefore AD + DB > AC$, or $AB > AC$, by Ax. 9. Q.E.D.

Given the triangle ABC, with AB greater than AC.

To prove that $\angle C$ *is greater than* $\angle B$.

Proof. The $\angle C$ must be equal to, less than, or greater than the $\angle B$.

If $\angle C = \angle B$, then $AB = AC$; § 680

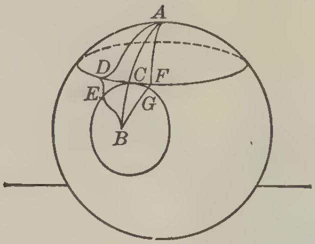
and if $\angle C$ is less than $\angle B$, then $AB < AC$, as above.

But both of these conclusions are contrary to what is given.

$\therefore \angle C$ is greater than $\angle B$. Q.E.D.

PROPOSITION XXII. THEOREM

682. *The shortest line that can be drawn on the surface of a sphere between two points is the arc of a great circle joining the two points, not greater than a semicircle.*



Given AB, the arc of a great circle, not greater than a semicircle, joining the points A and B.

To prove that AB is the shortest line that can be drawn on the surface joining A and B.

Proof. Let C be any point in AB.

With A and B as poles and AC and BC as polar distances, describe two arcs DCF and GCE.

The arcs DCF and GCE have only the point C in common. For if F is any other point in DCF, and if arcs of great circles AF and BF are drawn, then

$AF = AC$. § 636

But $AF + BF > AC + BC$. § 663

Take away AF from the left member of the inequality, and its equal AC from the right member.

Then $BF > BC$. Ax. 6

Therefore $BF > BG$, the equal of BC. Ax. 9

Hence F lies outside the circle whose pole is B, and the arcs DCF and GCE have only the point C in common.

Now let *ADEB* be any line from *A* to *B* on the surface of the sphere, which does not pass through *C*.

This line will cut the arcs *DCF* and *GCE* in separate points *D* and *E*; and if we revolve the line *AD* about *A* as a fixed point until *D* coincides with *C*, we shall have a line from *A* to *C* equal to the line *AD*.

In like manner, we can draw a line from *B* to *C* equal to the line *BE*.

Therefore a line can be drawn from *A* to *B* through *C* that is equal to the sum of the lines *AD* and *BE*, and hence is less than the line *ADEB* by the line *DE*.

Therefore no line which does not pass through *C* can be the shortest line from *A* to *B*.

Therefore the shortest line from *A* to *B* passes through *C*.

But *C* is any point in the arc *AB*.

Therefore the shortest line from *A* to *B* passes through every point of the arc *AB*, and consequently coincides with the arc *AB*.

Therefore the shortest line from *A* to *B* is the great-circle arc *AB*. Q.E.D.

EXERCISE 106

1. The three medians of a spherical triangle are concurrent.

2. To construct with a given radius a spherical surface that passes through three given points.

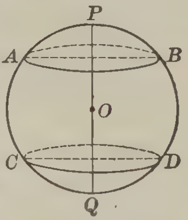
3. To construct with a given radius a spherical surface that passes through two given points and is tangent to a given plane.

4. To construct with a given radius a spherical surface that passes through two given points and is tangent to a given sphere.

5. The smallest circle on a given sphere whose plane passes through a given point within the sphere is the circle whose plane is perpendicular to the radius through the given point.

683. Zone. A portion of a spherical surface included between two parallel planes is called a *zone.*

Thus on the earth we have the torrid zone included between the planes of the tropics of Cancer and Capricorn.



The circles made by the planes are called the *bases* of the zone, and the distance between the planes is called the *altitude* of the zone.

If one of the planes is tangent to the sphere and the other plane cuts the sphere, the zone is called a *zone of one base.*

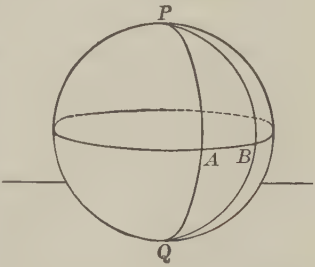
If both planes are tangent to the sphere, the zone is a complete spherical surface.

684. Generation of a Zone. If a great circle revolves about its diameter as an axis, any arc of the circle generates a zone.

Thus, in the figure of § 683, if the great circle $PACQ$ revolves about its diameter PQ as an axis, the arc AC generates the zone AD, of which the altitude is the distance between the parallel planes. Similarly, the arc AP generates the zone ABP, and the arc CQ generates the zone CDQ, these both being zones of one base.

685. Lune. A portion of a spherical surface bounded by the halves of two great circles is called a *lune.*

Thus $PAQB$ is a lune. A lune is evidently generated by the partial or complete revolution of half of a great circle about its diameter as an axis.



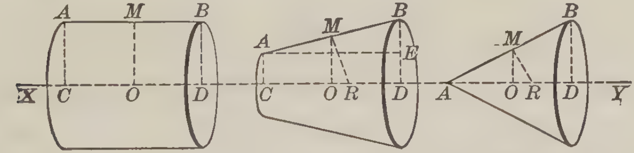
686. Angle of a Lune. The angle between the semicircles bounding a lune is called the *angle of the lune.*

Thus $\angle APB$ is the angle of the lune $PAQB$.

A lune is usually taken as having an angle less than a straight angle. This is not necessary, for we may consider a hemispherical surface as a lune with an angle of 180°. We may also conceive of lunes with angles greater than a straight angle, and we may even think of an entire spherical surface as a lune whose angle is 360°.

PROPOSITION XXIII. THEOREM

687. *The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis by the circle whose radius is a perpendicular erected at the mid-point of the line and terminated by the axis.*



Given an axis XY about which a line AB in the same plane with XY revolves, M being the mid-point of AB, CD being the projection of AB on XY, MO being perpendicular to XY, MR being perpendicular to AB, and a being the area generated by AB.

To prove that $a = CD \times 2\pi MR$.

Proof. 1. If AB is ∥ to XY, $CD = AB$, MR coincides with MO, and a is the lateral area of a right cylinder. § 588

2. If AB is not ∥ to XY, and does not cut XY, a is the lateral area of the frustum of a cone of revolution.

$\therefore a = AB \times 2\pi MO$. § 616

Draw AE ∥ to XY.

The $\triangle$ AEB and MOR are similar. § 290

$\therefore MO : AE = MR : AB$. § 282

$\therefore AB \times MO = AE \times MR$, § 261

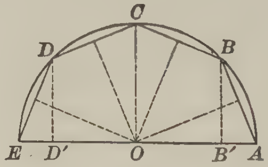
or $AB \times MO = CD \times MR$. Ax. 9

Substituting, $a = CD \times 2\pi MR$.

3. If A lies in the axis XY, then AE and CD coincide, and $a = CD \times 2\pi MR$, by § 609. Q.E.D.

PROPOSITION XXIV. THEOREM

688. *The area of the surface of a sphere is equal to the product of the diameter by the circumference of a great circle.*



Given a sphere generated by the semicircle *ABCDE* revolving about the diameter *AE* as an axis, *s* being the area of the surface, *r* being the radius, and *d* being the diameter.

To prove that $s = 2\pi rd$.

Proof. Inscribe in the semicircle half of a regular polygon having an even number of sides, as $ABCDE$.

From the center O draw ⊥s to the chords AB, BC, CD, DE.

These ⊥s bisect the chords (§ 174) and are equal. § 178

Let l denote the length of each of these ⊥s.

From B, C, and D drop perpendiculars to AE.

Then area of surface generated by $AB = AB' \times 2\pi l$, § 687

area of surface generated by $BC = B'O \times 2\pi l$, etc.

∴ area of surface generated by $ABCDE = AE \times 2\pi l$ Ax. 1

$= 2\pi ld$. Ax. 9

Denote the area of the surface generated by $ABCDE$ by s', and let the number of sides of $ABCDE$ be indefinitely increased.

Then s' approaches s as a limit,

l approaches r as a limit, § 377

and consequently $2\pi ld$ approaches $2\pi rd$ as a limit.

But $s' = 2\pi ld$, always. § 687

∴ $s = 2\pi rd$, by § 207. Q.E.D.

689. Corollary 1. *The area of the surface of a sphere is equivalent to the area of four great circles, or to* $4\pi r^2$.

In $s = 2\pi rd$, what is the value of d in terms of r? Then what is the value of s in terms of r?

For example, if the radius is 10 in., the area of the surface of the sphere is $4\pi \cdot 100$ sq. in., or 1256.64 sq. in.

690. Corollary 2. *The areas of the surfaces of two spheres are to each other as the squares on their radii, or as the squares on their diameters.*

If the radii are r and r', the diameters d and d', and the surfaces s and s', then what is the ratio of s to s', according to § 689? Show that this also equals $r^2 : r'^2$, and $d^2 : d'^2$.

691. Corollary 3. *The area of a zone is equal to the product of the altitude by the circumference of a great circle.*

If we apply the reasoning of § 688 to the zone generated by the revolution of the arc BCD, we obtain

$$\text{the area of zone } BCD = B'D' \times 2\pi r,$$

where $B'D'$ is the altitude of the zone and $2\pi r$ the circumference of a great circle.

For example, if the radius is 10 in., and the altitude is 5 in., the area of the zone is $5 \cdot 2\pi \cdot 10$ sq. in., or 314.16 sq. in.

692. Corollary 4. *The area of a zone of one base is equivalent to the area of a circle whose radius is the chord of the generating arc.*

The arc AB generates a zone of one base.

$$\therefore \text{the area of the zone } AB = AB' \times 2\pi r = \pi AB' \times AE.$$

But $$AB' \times AE = \overline{AB}^2. \qquad \text{§ 298}$$

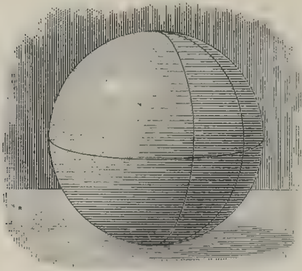
$$\therefore \text{the area of the zone } AB = \pi\overline{AB}^2.$$

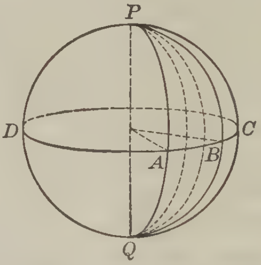
693. Spherical Excess of a Triangle. The excess of the sum of the angles of a spherical triangle over 180° is called the *spherical excess* of the triangle.

For example, if the angles of a spherical triangle are 80°, 90°, and 100°, the spherical excess of the triangle is 90°.

PROPOSITION XXV. THEOREM

694. *The area of a lune is to the area of the surface of the sphere as the angle of the lune is to four right angles.*





Given a lune *PAQB*, the great circle *ABCD* whose pole is *P*, *a* the value in degrees of the angle of the lune, *l* the area of the lune, and *s* the area of the surface of the sphere.

To prove that $l : s = a : 4$ *rt.* ∠s.

Proof. The arc AB measures the ∠ a of the lune. § 654

Hence arc AB : circle $ABCD = a : 4$ rt. ∠s. § 212

If AB and $ABCD$ are commensurable, let their common measure be contained m times in AB, and n times in $ABCD$.

Then arc AB : circle $ABCD = m : n$.

$\therefore a : 4$ rt. ∠s $= m : n$.

Pass an arc of a great circle through the poles P and Q and each point of division of $ABCD$.

These arcs will divide the entire surface into n equal lunes, of which the lune $PAQB$ will contain m.

$\therefore l : s = m : n$.

$\therefore l : s = a : 4$ rt. ∠s. Ax. 8

If AB and $ABCD$ are incommensurable, the theorem can be proved by the method of limits as in § 472. Q.E.D.

EXERCISE 107

Using $\pi = 3.1416$ *for all examples in this exercise, find the areas of spheres whose radii are as follows:*

1. 2 in.
2. 7 in.
3. $3\frac{1}{2}$ in.
4. $5\frac{7}{8}$ in.
5. 2 ft. 1 in.
6. 3 ft. 6 in.
7. 48.8 in.
8. 4000 mi.

Find the radii of spheres whose areas are as follows:

9. 12.5664 sq. in.
10. 50.2656 sq. in.
11. 1 sq. ft.
12. $100\,\pi$ sq. in.
13. s.
14. $4\,\pi^3$.

On a sphere whose radius is 20 in., find the areas of zones whose altitudes are as follows:

15. 2 in.
16. 3 in.
17. 7 in.
18. 10 in.
19. 1 ft.
20. $2\frac{1}{2}$ in.
21. 3.45 in.
22. 6.83 in.

On a sphere whose radius is 10 in., find the areas of lunes whose angles are as follows:

23. 30°.
24. 45°.
25. 90°.
26. 180°.
27. 22° 30′.
28. 7° 30′.
29. 52° 20′ 20″.
30. 48° 35′ 10″.

31. Two lunes on the same sphere or on equal spheres have the same ratio as their angles.

32. The area of a lune is equal to one ninetieth of the area of a great circle multiplied by the number of degrees in the angle of the lune.

33. Zones on the same sphere or on equal spheres are to each other as their altitudes.

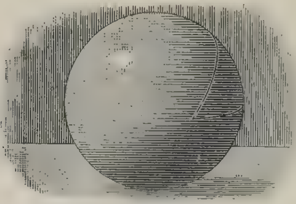
34. Given the radius of a sphere 15 in., find the area of a lune whose angle is 30°.

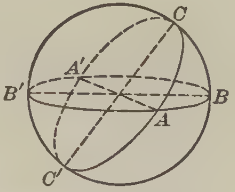
35. Given the diameter of a sphere 16 in., find the area of a lune whose angle is 75°.

36. What is the spherical excess of a trirectangular triangle?

PROPOSITION XXVI. THEOREM

695. *A spherical triangle is equivalent to a lune whose angle is half the spherical excess of the triangle.*





Given the spherical triangle *ABC* on a sphere of surface *s*.

To prove that $\triangle ABC$ *is equivalent to a lune whose angle is* $\frac{1}{2}(\angle A + \angle B + \angle C - 180°)$.

Proof. Produce the sides of the $\triangle ABC$ to complete circles.

Now $\triangle s$ $AB'C'$ and $A'BC$ are symmetric. Const.

$\therefore \triangle AB'C'$ is equivalent to $\triangle A'BC$. § 674

$\therefore$ lune $ABA'C = \triangle ABC + \triangle AB'C'$. Ax. 9

But $\triangle CB'A + \triangle AC'B + \triangle AB'C' + \triangle ABC = \frac{1}{2}s$. Ax. 11

$\therefore$ (lune $BCB'A - \triangle ABC$) + (lune $CAC'B - \triangle ABC$)
+ lune $ABA'C = \frac{1}{2}s$. Ax. 9

$\therefore 2\triangle ABC =$ lune $BCB'A$ + lune $CAC'B$
+ lune $ABA'C - \frac{1}{2}s$. Axs. 1, 2

$\therefore \triangle ABC = \frac{1}{2}$(lune $BCB'A$ + lune $CAC'B$
+ lune $ABA'C - \frac{1}{2}s$). Ax. 4

But $\frac{1}{2}s =$ a lune whose angle is 180°. § 694

$\therefore \triangle ABC =$ a lune whose angle is
$\frac{1}{2}(\angle A + \angle B + \angle C - 180°)$. Q.E.D.

Discussion. Since we have found (§ 694) how to compute the area of a lune, we can now compute the area of a spherical triangle when the angles are known.

696. Corollary. *If two great-circle arcs intersect within a great circle, the sum of the two opposite spherical triangles which they form with the great circle is equivalent to a lune whose angle is the angle between the arcs.*

697. Computation of Area. To illustrate the computation involved in § 695, find the area of a triangle whose angles are 110°, 100°, and 95°, on the surface of a sphere whose radius is 6 in.

$$\text{Spherical excess} = 110^\circ + 100^\circ + 95^\circ - 180^\circ = 125^\circ.$$

$$\therefore \text{ angle of lune} = 62\tfrac{1}{2}^\circ.$$

$$\therefore \text{ area of lune} = \frac{62\frac{1}{2}}{360} \text{ of the spherical surface.}$$

$$\therefore \text{ area of lune} = \frac{62\frac{1}{2}}{360} \times 4 \times 3.1416 \times 36 \text{ sq. in.}$$

$$\therefore \text{ area of triangle} = 78.54 \text{ sq. in.}$$

698. Spherical Excess of a Polygon. The excess of the sum of the angles of a spherical polygon of n sides over $(n-2) \times 180^\circ$ is called the *spherical excess* of the polygon.

EXERCISE 108

Compute the areas of triangles on spheres of the given diameters, the angles being as follows:

1. 100°, 120°, 140°, $d = 16$ in.
2. 105°, 130°, 125°, $d = 10$ in.
3. 127°, 132°, 90°, $d = 20$ in.
4. 115°, 124°, 85°, $d = 30$ in.
5. 135°, 110°, 92°, $d = 40$ in.
6. 148°, 93°, 68°, $d = 25.8$ in.
7. 115° 27′ 30″, 102° 32′ 48″, 68° 27′ 39″, $d = 8000$ mi.

Compute the areas of triangles on spheres of the given radii, the angles being as follows:

8. 120°, 100°, 90°, $r = 9$ in.
9. 130°, 90°, 80°, $r = 10$ in.
10. 105°, 75°, 65°, $r = 18$ in.
11. 115°, 102°, 30°, $r = 36$ in.
12. 140°, 120°, 85°, $r = 90$ in.
13. 136°, 117°, 93°, $r = 1.8$ in.

Compute the areas of triangles on spheres of the given circumferences, the angles being as follows:

14. 93°, 94°, 120°, $c = 31.416$ in.

15. 82°, 105°, 98°, $c = 62.832$ in.

16. 148°, 27°, 125°, $c = 15.708$ in.

17. 162°, 39°, 120°, $c = 78.54$ in.

18. 149°, 41°, 116°, $c = 39.27$ in.

19. $126° \; 30' \; 42''$, $105° \; 26' \; 15''$, $63° \; 15' \; 3''$, $c = 314.16$ in.

20. What is the area of a triangle on the earth's surface the vertices of which are the north pole and two points on the equator, one at 37° W. and the other at 16° E., the earth being considered a sphere with a radius of 4000 mi.?

21. If the radii of two spheres are 6 in. and 4 in. respectively, and the distance between the centers is 5 in., what is the area of the circle of intersection of the spheres?

22. Find the radius of the circle determined on a sphere of 5 in. diameter by a plane 1 in. from the center.

23. If the radii of two concentric spheres are r and r', and if a plane is passed tangent to the interior sphere, what is the area of the section made in the other sphere?

24. Two points A and B are 8 in. apart. Find the locus in space of a point 5 in. from A and 7 in. from B.

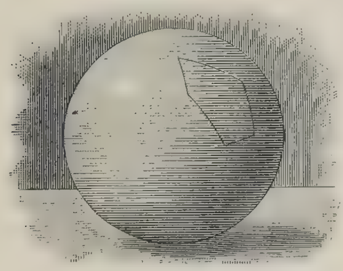
25. Two points A and B are 10 in. apart. Find the locus in space of a point 7 in. from A and 3 in. from B.

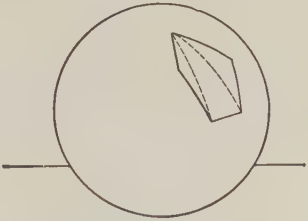
26. The radii of two parallel sections of the same sphere are a and b respectively, and the distance between the sections is d. Find the radius of the sphere.

27. The diameter of a certain sphere is $\sqrt{2}$. The chords of the arcs that form the sides of a triangle on the surface of the sphere are respectively 1, 1, and $\frac{1}{2}\sqrt{2}$. Find the area of the spherical triangle.

PROPOSITION XXVII. THEOREM

699. *A spherical polygon is equivalent to a lune whose angle is half the spherical excess of the polygon.*





Given a spherical polygon P of n sides, the sum of the angles being s.

To prove that P is equivalent to a lune whose angle is $\frac{1}{2}(s - \overline{n - 2} \times 180^\circ)$.

Proof. Draw all the diagonals from any vertex.

Since there is a distinct triangle for each side except those meeting at the vertex chosen, there are $(n - 2)$ triangles.

Since each triangle is equivalent to a lune whose angle is half the excess of the sum of its angles over 180°, § 695

therefore the $(n - 2)$ triangles are equivalent to a lune whose angle is half the excess of the sum of all the angles of the polygon over $(n - 2) \times 180^\circ$.

$\therefore P =$ a lune whose angle is $\frac{1}{2}(s - \overline{n - 2} \times 180^\circ)$. Q. E. D.

700. Computation of Area. Find the area of a spherical polygon whose angles are 100°, 110°, 120°, and 170°, r being 6 in.

$$\text{Spherical excess} = 100^\circ + 110^\circ + 120^\circ + 170^\circ - 2 \times 180^\circ = 140^\circ.$$

$$\therefore \text{angle of lune} = 70^\circ.$$

$$\begin{aligned} \therefore \text{area of lune} &= \tfrac{70}{360} \text{ of } 4\pi r^2 \\ &= \tfrac{7}{36} \text{ of } 4 \times 3.1416 \times 36 \text{ sq. in.} \\ &= 87.9648 \text{ sq. in.} \end{aligned}$$

EXERCISE 109

Find the areas of spherical polygons on spheres of the given areas, the angles being as follows:

1. 30°, 90°, 120°, 130°, $a = 2$ sq. ft.

2. 45°, 60°, 100°, 165°, $a = 288$ sq. in.

3. 70°, 168°, 92°, 120°, $a = 500$ sq. in.

4. 68° 30′, 149° 50′, 96° 54′, 136° 52′, $a = 750$ sq. in.

5. 122° 27′ 40″, 130° 32′ 50″, 98° 31′ 30″, 96° 48′, $a = 600$ sq. in.

6. 132°, 96°, 154°, 120°, 150°, $a = 3$ sq. ft. 120 sq. in.

7. 130°, 156°, 172°, 95°, 120°, 100°, $a = 157.2$ sq. in.

Find the areas of spherical polygons on spheres of the given radii, the angles being as follows:

8. 130°, 150°, 80°, 90°, $r = 10$ in.

9. 148°, 157°, 90°, 100°, 120°, $r = 20$ in.

10. 172°, 169°, 86°, 141°, 100°, 90°, $r = 24$ in.

11. 135° 30′, 148° 42′, 96° 37′, 102° 11′, $r = 10$ in.

Find the areas of spherical polygons on spheres of the given diameters, the angles being as follows:

12. 148°, 92°, 60°, 120°, $d = 10$ in.

13. 172°, 168°, 93°, 37°, 100°, $d = 22$ in.

14. 102°, 162°, 139°, 141°, 138°, 126°, $d = 20$ in.

15. 82° 50′ 42″, 120° 29′ 18″, 98° 37′ 15″, 141° 22′ 45″, $d = 20$ in.

Find the areas of spherical polygons on spheres of the given circumferences, the angles being as follows:

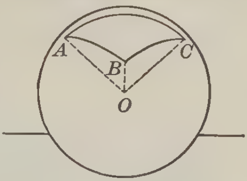
16. 39°, 148°, 172°, 168°, $c = 3.1416$ in.

17. 128°, 92°, 168°, 109°, $c = 31.416$ in.

18. 146°, 129°, 102°, 137°, 100°, $c = 6.2832$ in.

19. 128°, 145°, 139°, 82°, 161°, 137°, $c = 18.8496$ in.

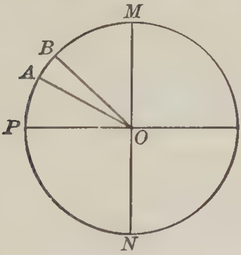
701. Spherical Pyramid. A portion of a sphere bounded by a spherical polygon and the planes of its sides is called a *spherical pyramid.*

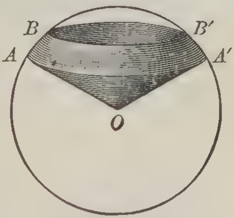


The center of the sphere is called the *vertex* of the spherical pyramid, and the spherical polygon is called the *base.*

Thus *O-ABC* is a spherical pyramid.

702. Spherical Sector. A portion of a sphere generated by the revolution of a circular sector about any diameter of the circle of which the sector is a part is called a *spherical sector.*





Thus if the sector *AOB* revolves about the diameter *MN* as an axis, it generates the spherical sector *AB-O-A′B′.*

The zone generated by the arc of the generating sector is called the *base* of the spherical sector.

703. Spherical Segment. A portion of a sphere contained between two parallel planes is called a *spherical segment.*

The sections of the sphere made by the parallel planes are called the *bases* of the spherical segment, and the distance between these bases is called the *altitude* of the spherical segment.

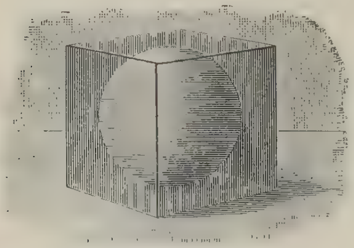
If one of the parallel planes is tangent to the sphere, the segment is called a *spherical segment of one base.*

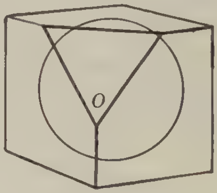
A spherical segment of one base may be generated by the revolution of a circular segment about the diameter perpendicular to its base.

704. Spherical Wedge. A portion of a sphere bounded by a lune and the planes of two great circles is called a *spherical wedge.*

PROPOSITION XXVIII. THEOREM

705. *The volume of a sphere is equal to the product of the area of its surface by one third of its radius.*





Given a sphere of radius r, area of surface s, volume v, and center O.

To prove that $v = s \times \frac{1}{3} r$.

Proof. We may imagine a cube of edge $2r$ circumscribed about the sphere.

Connect O with each of the vertices of this cube.

These connecting lines are the edges of six pyramids whose bases are the faces of the cube and whose altitudes all equal r.

The volume of each pyramid is a face of the cube multiplied by $\frac{1}{3} r$, and the volume of the six pyramids, or of the whole cube, is the area of the surface of the cube multiplied by $\frac{1}{3} r$.

Now imagine planes drawn tangent to the sphere, at the points where the edges of the pyramids cut its surface. We then have a circumscribed solid whose volume is nearer that of the sphere than is the volume of the circumscribed cube, but is still greater than the sphere. Ax. 11

Proceeding as before, connect O with the vertices of the new polyhedron. These connecting lines are the edges of pyramids whose bases are together equal to the faces of the polyhedron and whose common altitude is r. § 646

Then the sum of the volumes of these pyramids is again the area of the surface of the polyhedron multiplied by $\frac{1}{3}r$. Denoting this volume by v' and the area of the surface by s', we have

$$v' = s' \times \tfrac{1}{3}r.$$

If we continue to draw tangent planes to the sphere, we continue to diminish the circumscribed solid.

By continuing this process indefinitely we can make the difference between the volume of the sphere and the volume of the circumscribed solid less than any assigned positive quantity, however small, the difference between the surface of the sphere and the surface of the circumscribed solid becoming and remaining less than any assigned value, however small.

$\therefore v$ is the limit of v', and s is the limit of s'. § 204

And since it has been shown that

$$v' = s' \times \tfrac{1}{3}r, \text{ always,}$$

$$\therefore v = s \times \tfrac{1}{3}r, \text{ by § 207.}$$ Q.E.D.

706. Corollary 1. *The volume of a sphere of radius r and diameter d is equal to* $\frac{4}{3}\pi r^3$ *or* $\frac{1}{6}\pi d^3$.

For in $v = s \times \frac{1}{3}r$ what is the value of s in terms of r? What is the value of d in terms of r? Then what is the value of v in terms of d?

707. Corollary 2. *The volumes of two spheres are to each other as the cubes of their radii.*

What is the ratio of $\frac{4}{3}\pi r^3$ to $\frac{4}{3}\pi r'^3$?

By the same reasoning, the volumes are to each other as the cubes of the diameters.

708. Corollary 3. *The volume of a spherical sector is equal to one third the product of the area of the zone which forms its base multiplied by the radius of the sphere.*

Suppose the base divided into spherical triangles. The planes determined by their vertices are the bases of triangular pyramids with vertices at O. What is the limit of the sum of the volumes of these pyramids as the bases decrease in size?

EXERCISE 110

PROBLEMS OF COMPUTATION

Find the volumes of spheres whose radii are:

1. 3 in.
2. 5 in.
3. 7 in.
4. $2\frac{1}{2}$ in.
5. $4\frac{3}{8}$ in.
6. $9\frac{7}{8}$ in.
7. 20.7 ft.
8. 2 ft. 3 in.
9. 4000 mi.

Find the volumes of spheres whose diameters are:

10. 24 in.
11. 36 in.
12. 48 in.
13. 2.8 in.
14. 3.4 in.
15. 4.5 in.
16. 2 ft. 1 in.
17. 3 ft. 4 in.
18. 8 ft. 6 in.

Find the volumes of spheres whose circumferences are:

19. 6.2832 in.
20. 12.5664 in.
21. 18.8496 in.

Find the volumes of spheres whose surface areas are:

22. 12.5664 sq. in.
23. 50.2656 sq. in.
24. 113.0976 sq. in.

Find the radii of spheres whose volumes are:

25. 4.1888 cu. in.
26. 33.5104 cu. in.
27. 113.0976 cu. in.

28. The circumference of a hemispherical dome is 66 ft. How many square feet of lead are required to cover it?

29. If the ball on the top of St. Paul's Cathedral in London is 6 ft. in diameter, how much would it cost to gild it at 9 cents per square inch?

30. The dihedral angles made by the faces of a spherical pyramid are 80°, 100°, 120°, and 150°, and the length of a lateral edge is 42 ft. Find the area of the base.

31. The dihedral angles made by the faces of a spherical pyramid are 60°, 80°, and 100°, and the area of the base is 4π sq. ft. Find the radius.

32. What is the area of the surface of the earth?

Assume that the earth is a sphere with a radius of 4000 mi., and make the same assumption in subsequent examples relating to the earth.

33. The altitude of the torrid zone is 3200 mi. Find its area.

34. What is the area of the north temperate zone if its altitude is 1800 mi.?

35. Find the number of square miles of the earth's surface that can be seen from an aëroplane 1500 ft. above the surface.

36. How far in one direction can a man see from the deck of an ocean steamer if his eye is 40 ft. above the water?

37. To what height must a man be raised above the earth in order to see one sixth of its surface?

38. How much of the earth's surface would a man see if he were raised to the height of the radius above it?

39. If the atmosphere extends 50 mi. above the surface of the earth, find the volume of the atmosphere.

40. If an iron ball 4 in. in diameter weighs 9 lb., find the weight of a spherical iron shell 2 in. thick, the external diameter being 20 in.

41. What is the angle of a spherical wedge if its volume is $1\frac{1}{4}$ cu. ft. and the volume of the entire sphere is $8\frac{3}{4}$ cu. ft.?

42. The inside of a washbasin is in the shape of the segment of a sphere. The distance across the top is 16 in. and its greatest depth is 8 in. How many pints of water will it hold, allowing 7 gal. to the cubic foot?

43. Prove that the volume of a spherical pyramid is equal to the product of the base by one third of the radius, and find the volume if the base is one eighth of the surface of a sphere of radius 10 in.

44. Find the volume of a spherical sector whose base is a zone of area z, the radius of the sphere being r, following a process of reasoning similar to that in § 705.

EXERCISE 111

Formulas

1. Find the area z of the zone of a sphere of radius r, illuminated by a lamp placed at the height h above the surface.

2. Find the volume v of a sphere in terms of c, the circumference.

3. Find the radius r of a sphere in terms of v, the volume.

4. Find the diameter d of a sphere in terms of s, the area of the surface.

5. Find the circumference c of a sphere in terms of s, the area of the surface.

6. What is the altitude a of a zone, if its area is z and the volume of the sphere is v?

7. Show that in a spherical pyramid $v = \frac{1}{3}br$. Find r in terms of v and b; also b in terms of v and r.

8. Find a formula for the volume of the metal in a spherical iron shell, the inside radius being r and the thickness of the metal being t.

9. Find a formula for the weight of a spherical shell, the inside radius being r, the thickness of the metal being t, and the weight of a cubic unit of metal being w.

10. If the area of a zone z equals $2\pi ra$ (§ 691), find a formula for a in terms of z and r.

11. If the area of a zone is expressed by the formula $z = 2\pi ra$, what is the diameter of the sphere upon which a zone z has an altitude a?

12. Find the area z of a zone of altitude a on a sphere whose area of surface is s.

13. Find a formula for the area a of that part of the surface of a sphere of radius r seen from a point at a distance d above the surface.

EXERCISE 112

PROBLEMS OF LOCI

Find the locus of a point:

1. At a given distance from a given point.

2. At a given distance from a given straight line.

3. At a given distance from a given plane.

4. At a given distance from a given cylindric surface.

5. At a given distance from a given spherical surface.

6. Equidistant from two given points.

7. Equidistant from two given planes.

8. At a given distance from a given point and at another given distance from a given straight line.

9. At a given distance from a given point and at another given distance from a given plane.

10. At a given distance from a given point and equidistant from two other given points.

11. At a given distance from a given point and equidistant from two given planes.

Find one or more points:

12. At a distance d_1 from a given point, at a distance d_2 from a given straight line, and at a distance d_3 from a given plane.

13. At a distance d_1 from a given point, at a distance d_2 from a given plane, and equidistant from two other given planes.

14. Equidistant from two given points, equidistant from two given planes, and at a distance r from a given point.

15. Find the locus of the center of a sphere whose surface touches two given planes and passes through two given points that lie between the planes.

EXERCISE 113

Miscellaneous Exercises

1. The volume of a sphere is to the volume of the inscribed cube as π is to $\frac{2}{3}\sqrt{3}$.

2. The volume of a sphere is to the volume of the circumscribed cube as π is to 6.

3. Find the ratio of the volume of a cube inscribed in a sphere to that of a circumscribed cube.

4. Find the difference between the volumes of two cubes, one inscribed in a sphere of radius 10 in. and the other circumscribed about it.

5. The planes perpendicular to the three faces of a trihedral angle, and bisecting the face angles, meet in a straight line.

6. The planes that pass through the edges of a trihedral angle, and are perpendicular to the opposite faces, meet in a straight line.

7. The altitude of a regular tetrahedron is equal to the sum of four perpendiculars let fall from any point within the tetrahedron upon the four faces.

8. To cut a given tetrahedral angle by a plane so that the section shall be a parallelogram.

9. Compare the volumes of the solids generated by the revolution of a rectangle successively about two adjacent sides, the sides being a and b respectively.

10. Find the difference between the volume of a frustum of a pyramid and the volume of a prism each 24 ft. high, if the bases of the frustum are squares with sides 20 ft. and 16 ft. respectively, and the base of the prism is the section of the frustum parallel to the bases and midway between them.

11. To draw a line through the vertex of any trihedral angle, making equal angles with its edges.

12. The lines drawn from each vertex of a tetrahedron to the point of intersection of the medians of the opposite face all meet in a point called the *center of gravity* of the tetrahedron, which divides each line so that the ratio of the shorter segment to the whole line is $1:4$.

13. The lines joining the mid-points of the opposite edges of a tetrahedron all pass through the center of gravity and are bisected by it.

14. The plane which bisects a dihedral angle of a tetrahedron divides the opposite edge into segments proportional to the areas of the faces that include the dihedral angle.

15. To cut a given cube by a plane so that the section shall be a regular hexagon.

16. The volume of a right circular cylinder is equal to the product of the lateral area by half the radius.

17. The volume of a right circular cylinder is equal to the product of the area of the rectangle which generates it, by the length of the circumference generated by the point of intersection of the diagonals of the rectangle.

18. If the altitude of a right circular cylinder is equal to the diameter of the base, the volume is equal to the total area multiplied by a third of the radius.

19. The surface of a sphere is two thirds the total surface of the circumscribed cylinder.

20. The volume of a sphere is two thirds the volume of the circumscribed cylinder.

21. Given a sphere, a cylinder circumscribed about the sphere, and a cone of two nappes inscribed in the cylinder. If any two planes are drawn perpendicular to the axis of the three figures, the spherical segment between the planes is equivalent to the difference between the corresponding cylindric and conic segments.

EXERCISE 114

Review Questions

1. How is a sphere generated?

2. What are two tests of equality of spheres?

3. If a plane cuts a sphere, what figure is formed? Is the same true of a plane cutting a cone?

4. What is the test of equal circles on a given sphere?

5. What is a great circle of a sphere? Name four properties of great circles.

6. What is meant by a plane being tangent to a sphere? State any proposition concerning a tangent plane, and the corresponding proposition in plane geometry.

7. Complete this statement: A sphere may be inscribed in · · ·. State the corresponding proposition in plane geometry.

8. Complete this statement: A sphere may be circumscribed about · · ·. State the corresponding proposition in plane geometry.

9. Complete this statement: A spherical surface is determined by · · · points not in the same plane. State the corresponding proposition in plane geometry.

10. What is the limit of the sum of the sides of a spherical polygon? What are the limits of the sum of the angles of a spherical triangle?

11. What is a polar triangle? State two propositions relating to polar triangles.

12. What is meant by symmetric spherical triangles? State two propositions relating to such triangles.

13. State two propositions relating to congruent spherical triangles.

14. How is the area of a spherical triangle found? How is the area of a spherical polygon found?

APPENDIX

709. Subjects Treated. As with plane geometry, so with solid geometry, there are many topics that might be taken in addition to those given in any textbook. The theorems and problems already given in this work are standard propositions that are looked upon as basal, and are usually required as preliminary to more advanced work, and these, with a reasonable selection from the exercises, will be all that most schools have time to consider. It occasionally happens, however, that a school is able to do more than this, and then more exercises may be selected from the large number contained in this work, and a few additional topics may be studied. For this latter purpose the appendix is added, but its study should not be undertaken at the expense of good work on the fundamental propositions and the exercises depending upon them.

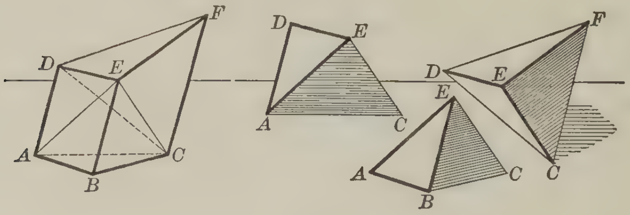
The subjects treated are certain additional propositions in the mensuration of solids, and a few general theorems relating to similar polyhedrons, these being occasionally required for college examinations. There is also added a brief sketch of the history of geometry, which all students are advised to read as a matter of general information, and a few of those recreations of geometry that add a peculiar interest to the subject.

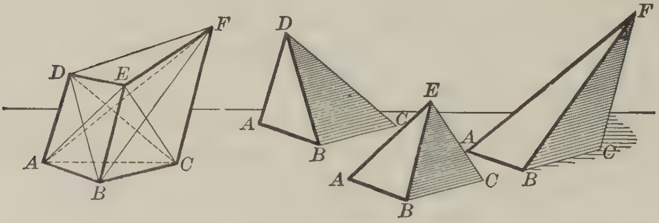
710. Similar Polyhedrons. Polyhedrons that have the same number of faces, respectively similar and similarly placed, and their corresponding polyhedral angles equal, are called *similar polyhedrons.*

It will be seen that this is analogous to the definition of similar polygons in plane geometry.

Proposition I. Theorem

711. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism and whose vertices are the three vertices of the inclined section.*





Given a truncated triangular prism *ABC-DEF* whose base is *ABC* and inclined section *DEF*, the truncated prism being divided into the three pyramids *E-ABC*, *E-ACD*, and *E-CFD*.

To prove ABC-DEF *equivalent to the sum of the three pyramids* E-ABC, D-ABC, *and* F-ABC.

Proof. E-ABC has the base ABC and the vertex E.

Now pyramid E-ACD = pyramid B-ACD. § 558

(*For they have the same base,* ACD, *and the same altitude, since their vertices* E *and* B *are in the line* $EB \parallel$ *to the plane* ACD.)

But the pyramid B-ACD may be regarded as having the base ABC and the vertex D; that is, as pyramid D-ABC.

Then since $\triangle$ *CFD* and *ACF* have the common base *CF* and equal altitudes, their vertices lying in the line *AD* which is parallel to *CF*, they are equivalent. § 326

Furthermore, pyramids *E-CFD* and *B-ACF* not only have equivalent bases, the $\triangle$ *CFD* and *ACF*, but they have the same altitude, since their vertices *E* and *B* are in the line *EB* which is parallel to the plane of their bases.

$\therefore$ pyramid *E-CFD* = pyramid *B-ACF*. § 558

But the pyramid *B-ACF* may be regarded as having the base *ABC* and the vertex *F*; that is, as pyramid *F-ABC*.

Therefore the truncated triangular prism *ABC-DEF* is equivalent to the sum of the three pyramids *E-ABC*, *D-ABC*, and *F-ABC*. Q.E.D.

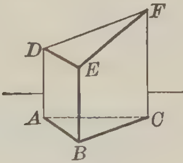


FIG. 1

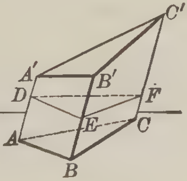


FIG. 2

712. COROLLARY 1. *The volume of a truncated right triangular prism is equal to the product of its base by one third the sum of its lateral edges.*

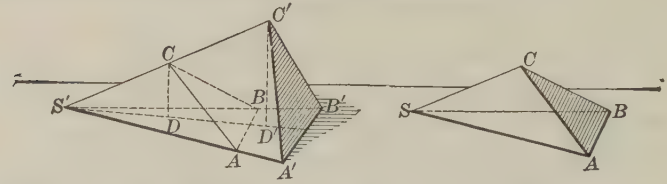
For the lateral edges *DA*, *EB*, *FC* (Fig. 1), being perpendicular to the base *ABC*, are the altitudes of the three pyramids whose sum is equivalent to the truncated prism. It is interesting to consider the special case in which $\triangle$ *DEF* is parallel to $\triangle$ *ABC*.

713. COROLLARY 2. *The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.*

For the right section *DEF* (Fig. 2) divides the truncated prism into two truncated right prisms.

Proposition II. Theorem

714. *The volumes of two tetrahedrons that have a trihedral angle of the one equal to a trihedral angle of the other are to each other as the products of the three edges of these trihedral angles.*



Given the two tetrahedrons S-ABC and S'-$A'B'C'$, having the trihedral angles S and S' equal, v and v' denoting the volumes.

To prove that $\dfrac{v}{v'} = \dfrac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'}$.

Proof. Place the tetrahedron S-ABC upon S'-$A'B'C'$ so that the trihedral $\angle S$ shall coincide with the equal trihedral $\angle S'$.

Draw CD and $C'D' \perp$ to the plane $S'A'B'$,

and let their plane intersect $S'A'B'$ in $S'DD'$.

The faces $S'AB$ and $S'A'B'$ may be taken as the bases, and CD, $C'D'$ as the altitudes, of the triangular pyramids C-$S'AB$ and C'-$S'A'B'$ respectively.

Then $\dfrac{v}{v'} = \dfrac{S'AB \times CD}{S'A'B' \times C'D'} = \dfrac{S'AB}{S'A'B'} \times \dfrac{CD}{C'D'}$. § 562

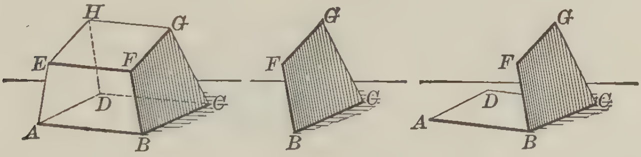
But $\dfrac{S'AB}{S'A'B'} = \dfrac{S'A \times S'B}{S'A' \times S'B'}$, § 332

and $\dfrac{CD}{C'D'} = \dfrac{S'C}{S'C'}$. § 282

$\therefore \dfrac{v}{v'} = \dfrac{S'A \times S'B \times S'C}{S'A' \times S'B' \times S'C'} = \dfrac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'}$, by Ax. 9. Q.E.D.

PROPOSITION III. THEOREM

715. *In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.*



Given the polyhedron AG, e denoting the number of edges, v the number of vertices, and f the number of faces.

To prove that $e + 2 = v + f$.

Proof. Beginning with one face $BCGF$, we have $e = v$.

Annex a second face $ABCD$ by applying one of its edges to a corresponding edge of the first face, and there is formed a surface of two faces having *one* edge BC and *two* vertices B and C common to the two faces.

Therefore for two faces $e = v + 1$.

Annex a third face $ABFE$, adjoining each of the first two faces. This face will have *two* edges AB, BF and *three* vertices A, B, F in common with the surface already formed.

Therefore for three faces $e = v + 2$.

In like manner, for four faces, $e = v + 3$, and so on.

Therefore for $(f - 1)$ faces $e = v + (f - 2)$.

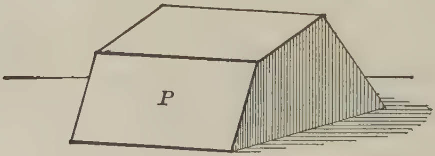
But $f - 1$ is the number of faces of the polyhedron when only one face is lacking, and the addition of this face will not increase the number of edges or vertices. Hence for f faces

$$e = v + f - 2, \text{ or } e + 2 = v + f. \qquad \text{Q.E.D.}$$

This theorem is due to the great Swiss mathematician, Euler.

PROPOSITION IV. THEOREM

716. *The sum of the face angles of any polyhedron is equal to four right angles taken as many times, less two, as the polyhedron has vertices.*



Given the polyhedron P, e denoting the number of edges, v the number of vertices, f the number of faces, and s the sum of the face angles.

To prove that $s = (v - 2)\, 4$ *rt.* $\measuredangle$s.

Proof. Since e denotes the number of edges, $2e$ will denote the number of sides of the faces, considered as independent polygons, for each edge is common to two polygons.

If an exterior angle is formed at each vertex of every polygon, the sum of the interior and exterior angles at each vertex is 2 rt. $\measuredangle$s; and since there are $2e$ vertices, the sum of the interior and exterior angles of all the faces is

$$2e \times 2 \text{ rt. } \measuredangle\text{s, or } e \times 4 \text{ rt. } \measuredangle\text{s.}$$

But the sum of the ext. $\measuredangle$s of each face is 4 rt. $\measuredangle$s. § 146

Therefore the sum of all the ext. $\measuredangle$s of f faces is

$$f \times 4 \text{ rt. } \measuredangle\text{s.}$$

Therefore $s = e \times 4 \text{ rt. } \measuredangle\text{s} - f \times 4 \text{ rt. } \measuredangle\text{s}$

$= (e - f)\, 4 \text{ rt. } \measuredangle\text{s.}$

But $e + 2 = v + f$; § 715

that is, $e - f = v - 2$. Ax. 2

Therefore $s = (v - 2)\, 4 \text{ rt. } \measuredangle\text{s.}$ Q.E.D.

EXERCISE 115

Find the volumes of truncated triangular prisms, given the bases b, and the distances of the three vertices p, q, r from the planes of the bases, as follows:

1. $b = 8$ sq. in., $p = 3$ in., $q = 4$ in., $r = 5$ in.
2. $b = 9$ sq. in., $p = 6$ in., $q = 3$ in., $r = 4\frac{1}{2}$ in.
3. $b = 15$ sq. in., $p = 7$ in., $q = 9$ in., $r = 8.1$ in.
4. $b = 32$ sq. in., $p = 9$ in., $q = 12$ in., $r = 9.3$ in.
5. $b = 48$ sq. in., $p = 16$ in., $q = 15$ in., $r = 18$ in.

6. A triangular rod of iron is cut square off (i.e. in right section) at one end, and slanting at the other end. The right section is an equilateral triangle $1\frac{1}{2}$ in. on a side. The edges of the rod are 3 ft. 2 in., 3 ft. 3 in., and 3 ft. 3 in. Find the weight of the rod, allowing 0.28 lb. per cubic inch.

7. Two triangular pyramids with a trihedral angle of the one equal to a trihedral angle of the other have the edges of these angles 3 in., 4 in., $3\frac{1}{2}$ in., and 5 in., $5\frac{1}{2}$ in., 6 in. respectively. Find the ratio of the volumes.

8. Make a table giving the number of edges, vertices, and faces of each of the five regular polyhedrons, showing that in every case the number conforms to Euler's theorem (§ 715).

9. Make a table similar to that of Ex. 8, giving the sum of the face angles in each of the five regular polyhedrons, showing that in every case $s = (v - 2)\ 4$ rt. $\angle s$ (§ 716).

10. There can be no seven-edged polyhedron.

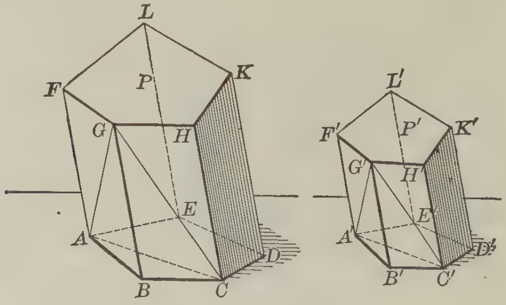
11. Can there be a nine-edged polyhedron?

12. What is the sum of the face angles of a six-edged polyhedron?

13. What is the sum of the face angles of a polyhedron with five vertices? with four vertices? Consider the possibility of a polyhedron with three vertices.

PROPOSITION V. THEOREM

717. *Two similar polyhedrons can be separated into the same number of tetrahedrons similar each to each and similarly placed.*



Given two similar polyhedrons P and P'.

To prove that P and P' can be separated into the same number of tetrahedrons similar each to each and similarly placed.

Proof. Let G and G' be corresponding vertices.

Divide all the faces of P and P', except those which include the angles G and G', into corresponding triangles by drawing corresponding diagonals.

Pass a plane through G and each diagonal of the faces of P; also pass a plane through G' and each corresponding diagonal of P'.

Any two corresponding tetrahedrons $G\text{-}ABC$ and $G'\text{-}A'B'C'$ have the faces ABC, GAB, GBC similar respectively to the faces $A'B'C'$, $G'A'B'$, $G'B'C'$. § 292

Since $\frac{AG}{A'G'} = \frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{GC}{G'C'}$, § 282

$\therefore$ the face GAC is similar to the face $G'A'C'$. § 289

They also have the corresponding trihedral ∡s equal. § 498

∴ the tetrahedron $G\text{-}ABC$ is similar to $G'\text{-}A'B'C'$. § 710

If $G\text{-}ABC$ and $G'\text{-}A'B'C'$ are removed, the polyhedrons remaining continue similar; for the new faces GAC and $G'A'C'$ have just been proved similar, and the modified faces AGF and $A'G'F'$, GCH and $G'C'H'$, are similar (§ 292); also the modified polyhedral ∡s G and G', A and A', C and C' remain equal each to each, since the corresponding parts taken from these angles are equal.

The process of removing similar tetrahedrons can be carried on until the polyhedrons are separated into the same number of tetrahedrons similar each to each and similarly placed. Q.E.D.

718. Corollary 1. *The corresponding edges of similar polyhedrons are proportional.*

For the corresponding faces are similar. Therefore their corresponding sides are proportional (§ 282).

719. Corollary 2. *Any two corresponding lines in two similar polyhedrons have the same ratio as any two corresponding edges.*

For these lines may be shown to be sides of similar polygons, and hence § 282 applies.

720. Corollary 3. *Two corresponding faces of similar polyhedrons are proportional to the squares on any two corresponding edges.*

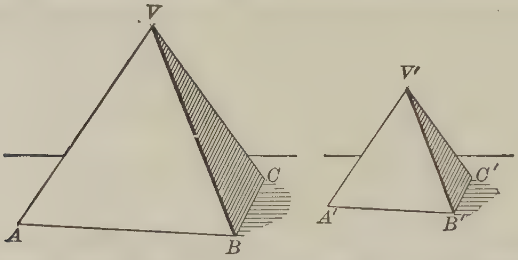
For they are similar polygons, and hence they are to each other as the squares on any two corresponding sides (§ 334).

721. Corollary 4. *The entire surfaces of two similar polyhedrons are proportional to the squares on any two corresponding edges.*

For the corresponding faces are proportional to the squares on any two corresponding edges (§ 720), and hence their sum has the same proportion, by § 269.

PROPOSITION VI. THEOREM

722. *The volumes of two similar tetrahedrons are to each other as the cubes on any two corresponding edges.*



Given two similar tetrahedrons $V\text{-}ABC$ and $V'\text{-}A'B'C'$, with volumes v and v', VB and $V'B'$ being two corresponding edges.

To prove that $$\frac{v}{v'} = \frac{\overline{VB}^3}{\overline{V'B'}^3}.$$

Proof. Since the two polyhedrons are similar, Given

$\therefore$ the corresponding polyhedral angles are equal, § 710

and, in particular, the trihedral angles V and V' are equal.

$$\therefore \frac{v}{v'} = \frac{VB \times VC \times VA}{V'B' \times V'C' \times V'A'} \qquad \text{§ 714}$$

$$= \frac{VB}{V'B'} \times \frac{VC}{V'C'} \times \frac{VA}{V'A'}.$$

Furthermore, since the tetrahedrons are similar, Given

$$\therefore \frac{VB}{V'B'} = \frac{VC}{V'C'} = \frac{VA}{V'A'}. \qquad \text{§ 718}$$

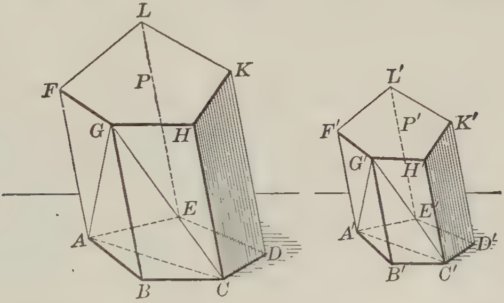
Substituting $\frac{VB}{V'B'}$ for its equals, we have

$$\frac{v}{v'} = \frac{VB}{V'B'} \times \frac{VB}{V'B'} \times \frac{VB}{V'B'}, \qquad \text{Ax. 9}$$

or $$\frac{v}{v'} = \frac{\overline{VB}^3}{\overline{V'B'}^3}. \qquad \text{Q.E.D}$$

PROPOSITION VII. THEOREM

723. *The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges.*



Given two similar polyhedrons P and P', with volumes v and v', GB and $G'B'$ being any two corresponding edges.

To prove that $v : v' = \overline{GB}^3 : \overline{G'B'}^3.$

Proof. Separate P and P' into tetrahedrons similar each to each and similarly placed (§ 717), denoting their respective volumes by $v_1, v_2, v_3, \cdots, v_1', v_2', v_3', \cdots$.

Then since $v_1 : v_1' = \overline{GB}^3 : \overline{G'B'}^3,$

$v_2 : v_2' = \overline{GB}^3 : \overline{G'B'}^3$, and so on. § 722

$\therefore v_1 + v_2 + v_3 + \cdots : v_1' + v_2' + v_3' + \cdots = \overline{GB}^3 : \overline{G'B'}^3.$ § 269

But $v_1 + v_2 + v_3 + \cdots = v$, and $v_1' + v_2' + v_3' + \cdots = v'$.

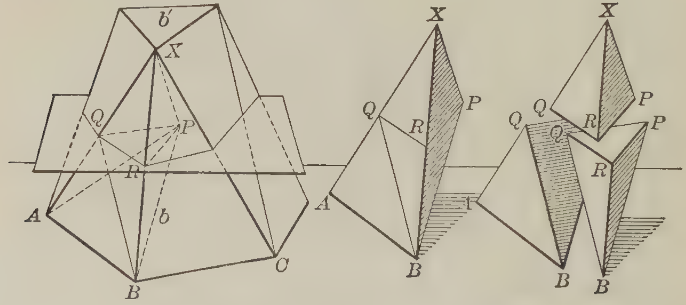
$\therefore v : v' = \overline{GB}^3 : \overline{G'B'}^3$, by Ax. 9. Q.E.D.

724. Prismatoid. A polyhedron having for bases two polygons in parallel planes, and for lateral faces triangles or trapezoids with one side common with one base, and the opposite vertex or side common with the other base, is called a *prismatoid*.

The *altitude* is the distance between the planes of the bases. The *mid-section* is the section made by a plane parallel to the bases and bisecting the altitude.

Proposition VIII. Theorem

725. *The volume of a prismatoid is equal to the product of one sixth of its altitude into the sum of its bases and four times its mid-section.*



Given a prismatoid of volume v, bases b and b', mid-section m, and altitude a.

To prove that $v = \frac{1}{6} a (b + b' + 4m)$.

Proof. If any lateral face is a trapezoid, divide it into two triangles by a diagonal.

Take any point P in the mid-section and join P to the vertices of the polyhedron and of the mid-section.

Separate the prismatoid into pyramids which have their vertices at P, and for their respective bases the lower base b, the upper base b', and the lateral faces of the prismatoid.

The pyramid P-XAB, which we may call a lateral pyramid, is composed of the three pyramids P-XQR, P-QBR, and P-QAB.

Now P-XQR may be regarded as having vertex X and base PQR, and P-QBR as having vertex B and base PQR.

Hence the volume of P-XQR is equal to $\frac{1}{6} a \cdot PQR$,

and the volume of P-QBR is equal to $\frac{1}{6} a \cdot PQR$. § 559

The pyramids P-QAB and P-QBR have the same vertex P. The base QAB is twice the base QBR (§ 327), since the $\triangle\, QAB$ has its base AB twice the base QR of the $\triangle\, QBR$ (§ 136), and these triangles have the same altitude (§ 724).

Hence the pyramid P-QAB is equivalent to twice the pyramid P-QBR. § 563

Hence the volume of P-QAB is equal to $\frac{2}{6}\, a \cdot PQR$.

Therefore the volume of P-XAB, which is composed of P-XQR, P-QBR, and P-QAB, is equal to $\frac{4}{6}\, a \cdot PQR$.

In like manner, the volume of each lateral pyramid is equal to $\frac{4}{6}\, a \times$ the area of that part of the mid-section which is included within it; and therefore the total volume of all these lateral pyramids is equal to $\frac{4}{6}\, am$.

The volume of the pyramid with base b is $\frac{1}{6}\, ab$,
and the volume of the pyramid with base b' is $\frac{1}{6}\, ab'$. § 559

Therefore $v = \frac{1}{6}\, a\,(b + b' + 4\, m)$. Q.E.D.

EXERCISE 116

Deduce from the formula for the volume of a prismatoid, $v = \frac{1}{6}\, a\,(b + b' + 4\, m)$, *the following formulas:*

1. Cube, $v = a^3$.

2. Prism, $v = ba$.

3. Pyramid, $v = \frac{1}{3}\, ba$.

4. Parallelepiped, $v = ba$.

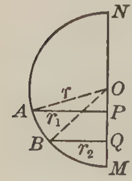
5. Frustum of a pyramid, $v = \frac{1}{3}\, a\left(b + b' + \sqrt{bb'}\right)$.

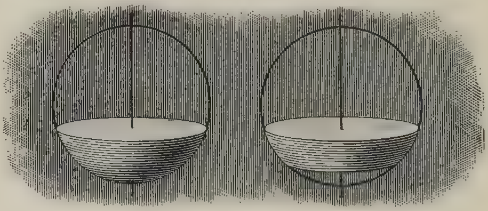
6. A prismatoid has an upper base 3 sq. in., a lower base 7 sq. in., an altitude 3 in., and a mid-section 4 sq. in. What is the volume?

7. A wedge has for its base a rectangle l in. long and w in. wide. The cutting edge is e in. long, and is parallel to the base. The distance from e to the base is d in. Deduce a formula for the volume of the wedge. Apply this formula to the case in which $l = 6$, $w = 1$, $e = 5$, $d = 3$.

Proposition IX. Theorem

726. *The volume of a spherical segment is equal to the product of one half the sum of its bases by its altitude, increased by the volume of a sphere having that altitude for its diameter.*





Given a spherical segment of volume v, generated by the revolution of $ABQP$ about MN as an axis, r being the radius of the sphere, AP being represented by r_1, BQ by r_2, and PQ by a.

To prove that $v = \frac{1}{2} a(\pi r_1^2 + \pi r_2^2) + \frac{1}{6}\pi a^3$.

Proof. We shall first find the volume of the spherical segment with one base, generated by AMP.

Area of zone $AM = 2\pi r \cdot PM$. § 691

$\therefore$ volume of sector generated by $OAM = \frac{1}{3} r \times 2\pi r \cdot PM$. § 708

But the cone generated by $OAP = \frac{1}{3}\pi r_1^2 (r - PM)$. § 611

$\therefore$ volume $AMP = \frac{1}{3} r \times 2\pi r \cdot PM \cdot - \frac{1}{3}\pi r_1^2 (r - PM)$. Ax. 2

But $r_1^2 = PM \times NP = PM(2r - PM)$. § 297

$\therefore$ volume $AMP = \frac{1}{3} r \times 2\pi r \cdot PM$

$- \frac{1}{3}\pi \cdot PM(2r - PM)(r - PM)$ Ax. 9

$= \pi \cdot \overline{PM}^2 (r - \frac{1}{3} PM)$.

In the same way, volume $BMQ = \pi \cdot \overline{QM}^2 (r - \frac{1}{3} QM)$.

$\therefore v =$ volume $AMP -$ volume BMQ

$= \pi \cdot \overline{PM}^2 \cdot r - \frac{1}{3}\pi \cdot \overline{PM}^3 - \pi \cdot \overline{QM}^2 \cdot r + \frac{1}{3}\pi \cdot \overline{QM}^3$

$= \pi r(\overline{PM}^2 - \overline{QM}^2) - \frac{1}{3}\pi(\overline{PM}^3 - \overline{QM}^3)$.

But $PM - QM = a$. Given

$\therefore v = \pi ra(PM + QM) - \frac{1}{3}\pi a(\overline{PM}^2 + PM \cdot QM + \overline{QM}^2)$. Ax. 9

But $a^2 = \overline{PM}^2 - 2\,PM \cdot QM + \overline{QM}^2$. Ax. 5

$\therefore a^2 + 3\,PM \cdot QM = \overline{PM}^2 + PM \cdot QM + \overline{QM}^2$. Ax. 1

$\therefore v = \pi ra(PM + QM) - \frac{1}{3}\pi a(a^2 + 3\,PM \cdot QM)$. Ax. 9

Furthermore $(2r - PM)\,PM = r_1^2$,

and $(2r - QM)\,QM = r_2^2$. § 297

$\therefore 2r \cdot PM + 2r \cdot QM - \overline{PM}^2 - \overline{QM}^2 = r_1^2 + r_2^2$. Ax. 1

$\therefore r \cdot PM + r \cdot QM = \frac{r_1^2 + r_2^2}{2} + \frac{\overline{PM}^2 + \overline{QM}^2}{2}$. Axs. 1, 4

$$\therefore v = \pi a\left(\frac{r_1^2 + r_2^2}{2} + \frac{\overline{PM}^2 + \overline{QM}^2}{2} - \frac{a^2}{3} - PM \cdot QM\right)$$

$$= \pi a\left(\frac{r_1^2 + r_2^2}{2} + \frac{a^2}{2} + PM \cdot QM - \frac{a^2}{3} - PM \cdot QM\right)$$

$$= \tfrac{1}{2}a(\pi r_1^2 + \pi r_2^2) + \tfrac{1}{6}\pi a^3.$$ Q.E.D.

EXERCISE 117

Find the volumes of spherical segments having bases b and b′, and altitudes a, as follows:

1. $b = 4$, $b' = 5$, $a = 1$.
2. $b = 4$, $b' = 6$, $a = 1\frac{1}{4}$.
3. $b = 5$, $b' = 7$, $a = 2\frac{1}{8}$.
4. $b = 6$, $b' = 8$, $a = 1\frac{1}{2}$.
5. $b = 8$, $b' = 12$, $a = 2$.
6. $b = 12$, $b' = 15$, $a = 3\frac{1}{2}$.
7. $b = 27$ sq. in., $b' = 32$ sq. in., $a = 2.33$ in.

Find the volumes of spherical segments having radii of bases r_1 and r_2, and altitudes a, as follows:

8. $r_1 = 3$, $r_2 = 4$, $a = 2$.
9. $r_1 = 4$, $r_2 = 7$, $a = 3$.
10. $r_1 = 8$, $r_2 = 5$, $a = 4\frac{1}{2}$.
11. $r_1 = 5$, $r_2 = 3$, $a = 1\frac{1}{2}$.
12. $r_1 = 6$, $r_2 = 5$, $a = 1\frac{1}{4}$.
13. $r_1 = 9$, $r_2 = 10$, $a = 2\frac{3}{4}$.
14. $r_1 = 9$ in., $r_2 = 7$ in., $a = 4.75$ in.

EXERCISE 118

EXAMINATION QUESTIONS

1. A pyramid 6 ft. high is cut by a plane parallel to the base, the area of the section being $\frac{1}{9}$ that of the base. How far from the vertex is the cutting plane?

2. Find the area of a spherical triangle whose angles are 100°, 120°, and 140°, the diameter of the sphere being 16 in.

3. Two angles of a spherical triangle are 80° and 120°. Find the limits of the third angle, and prove that the greatest possible area of the triangle is four times the least possible area, the sphere on which it is drawn being given.

4. An irregular portion, less than half, of a material sphere is given. Show how the radius can be found, compasses and ruler being allowed.

5. Find the volume of a cone of revolution, the area of the total surface of which is 200π sq. ft., and the altitude of which is 16 ft.

6. The volumes of two similar polyhedrons are 64 cu. ft. and 216 cu. ft. respectively. If the area of the surface of the first polyhedron is 112 sq. ft., find the area of the surface of the second polyhedron.

7. A solid sphere of metal of radius 12 in. is recast into a hollow sphere. If the cavity is spherical, of the same radius as the original sphere, find the thickness of the shell.

8. The stone spire of a church is a regular pyramid 50 ft. high on a hexagonal base each side of which is 10 ft. There is a hollow part which is also a regular pyramid 45 ft. high, on a hexagonal base of which each side is 9 ft. Find the number of cubic feet of stone in the spire.

9. The volumes of a hemisphere, right circular cone, and right circular cylinder are equal. Their bases are also equal, each being a circle of radius 10 in. Find the altitude of each.

10. A sphere of radius 5 ft. and a right circular cone also of radius 5 ft. stand on a plane. If the height of the cone is equal to a diameter of the sphere, find the position of the plane that cuts the two solids in equal circular sections.

11. The vertices of one regular tetrahedron are at the centers of the faces of another regular tetrahedron. Find the ratio of the volumes.

12. Find the area of a spherical triangle, if the perimeter of its polar triangle is 297° and the radius of the sphere is 10 centimeters.

13. The radii of two spheres are 13 in. and 15 in. respectively, and the distance between the centers is 14 in. Find the volume of the solid common to both spheres, — a spherical lens.

14. The radius of the base of a right circular cylinder is r and the altitude of the cylinder is a. Find the radius and the volume of a sphere whose surface is equivalent to the lateral surface of the cylinder.

15. If the polyhedral angle at the vertex of a triangular pyramid is trirectangular, and the areas of the lateral faces are a, b, and c respectively, and the area of the base is d, then $a^2 + b^2 + c^2 = d^2$.

16. If the earth is a sphere with a diameter of 8000 mi., find the area of the zone bounded by the parallels 30° north latitude and 30° south latitude. Show that this zone and the planes of the circles include $\frac{11}{16}$ of the volume of the earth.

17. The altitude of a cone of revolution is 12 centimeters and the radius of its base is 5 centimeters. Compute the radius of the sector of paper which, when rolled up, will just cover the convex surface of the cone, and compute the size of the central angle of this sector in degrees, minutes, and seconds.

18. The volume of any regular pyramid is equal to one third of its lateral area multiplied by the perpendicular distance from the center of its base to any lateral face.

19. If the area of a zone of one base is n times the area of the circle which forms its base, the altitude of the zone is $\frac{1}{n}(n-1)$ times the diameter of the sphere. Discuss the special case when $n=1$.

20. If the four sides of a spherical quadrilateral are equal, its diagonals are perpendicular to each other.

21. Find the volume of a pyramid whose base contains 30 square centimeters if one lateral edge is 5 centimeters and the angle formed by this edge and the plane of the base is 45°.

22. On the base of a right circular cone a hemisphere is constructed outside the cone. The surface of the hemisphere equals the surface of the cone. If r is the radius of the hemisphere, find the slant height of the cone, the inclination of the slant height to the base, and the volume of the entire solid.

23. Find the total surface and the volume of a regular tetrahedron whose edge equals 8 centimeters.

24. If a spherical quadrilateral is inscribed in a small circle, the sum of two opposite angles is equal to the sum of the other two angles.

25. By what number must the dimensions of a cylinder of revolution be multiplied to obtain a similar cylinder of revolution with surface n times that of the first? with volume n times that of the first?

26. A pyramid is cut by a plane parallel to the base midway between the vertex and the plane of the base. Compare the volumes of the entire pyramid and the pyramid cut off.

27. The height of a regular hexagonal pyramid is 36 ft. and one side of the base is 6 ft. What are the dimensions of a similar pyramid whose volume is $\frac{1}{20}$ that of the first?

28. One of the lateral edges of a pyramid is 4 meters. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts?

727. Recreations of Geometry. The following simple puzzles and recreations of geometry may serve the double purpose of adding interest to the study of the subject and of leading the student to exercise greater care in his demonstrations. They have long been used for this purpose and are among the best known puzzles of geometry.

EXERCISE 119

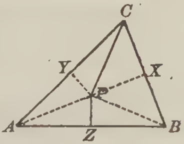
1. To prove that every triangle is isosceles.

Let ABC be a $\triangle$ that is not isosceles.

Take CP the bisector of $\angle ACB$, and ZP the $\perp$ bisector of AB.

These lines must meet, as at P, for otherwise they would be $\parallel$, which would require CP to be $\perp$ to AB, and this could happen only if $\triangle ABC$ were isosceles, which is not the case by hypothesis.

From P draw $PX \perp$ to BC and $PY \perp$ to CA, and draw PA and PB.



Then since ZP is the $\perp$ bisector of AB, $\therefore PA = PB$.

And since CP is the bisector of $\angle ACB$, $\therefore PX = PY$.

$\therefore$ the rt. $\triangle$s PBX and PAY are congruent, and $BX = AY$.

But the rt. $\triangle$s PXC and PYC are also congruent, and $\therefore XC = YC$.

Adding, we have $BX + XC = AY + YC$, or $BC = AC$.

$\therefore \triangle ABC$ is isosceles, even though constructed as not isosceles.

2. To prove that part of an angle equals the whole angle.

Take a square $ABCD$, and draw $MM'P$, the $\perp$ bisector of CD. Then $MM'P$ is also the $\perp$ bisector of AB.

From B draw any line BX equal to AB.

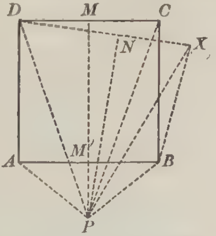
Draw DX and bisect it by the $\perp$ NP.

Since DX intersects CD, $\perp$s to these lines cannot be parallel, but must meet as at P.

Draw PA, PD, PC, PX, and PB.

Since MP is the $\perp$ bisector of CD, $PD = PC$. Similarly $PA = PB$, and $PD = PX$.

$\therefore PX = PD = PC$.

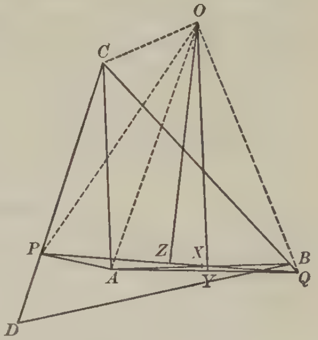


But $BX = BC$ by construction, and PB is common to $\triangle$s PBX and PBC.

$\therefore \triangle PBX$ is congruent to $\triangle PBC$, and $\angle XBP = \angle CBP$.

$\therefore$ the whole $\angle XBP$ equals its part, the $\angle CBP$.

3. To prove that part of an angle equals the whole angle.



Take a right triangle ABC and construct upon the hypotenuse BC an equilateral triangle BCD, as shown.

On CD lay off CP equal to CA.

Through X, the mid-point of AB, draw PX to meet CB produced at Q. Draw QA.

Draw the ⊥ bisectors of QA and QP, as YO and ZO. These must meet at some point O because they are ⊥ to two intersecting lines.

Draw OQ, OA, OP, and OC.

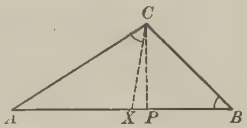
Since O is on the ⊥ bisector of QA, $\therefore OQ = OA$.

Similarly $OQ = OP$, and $\therefore OA = OP$.

But $CA = CP$, by construction, and $CO = CO$.

$\therefore \triangle AOC$ is congruent to $\triangle POC$, and $\angle ACO = \angle PCO$.

4. To prove that part of a line equals the whole line.



Take a triangle ABC and draw $CP \perp$ to AB.

From C draw CX, making $\angle ACX = \angle B$.

Then ▲ ABC and ACX are similar.

$\therefore \triangle ABC : \triangle ACX = \overline{BC}^2 : \overline{CX}^2$.

Furthermore $\triangle ABC : \triangle ACX = AB : AX$.

$$\therefore \overline{BC}^2 : \overline{CX}^2 = AB : AX,$$

or
$$\overline{BC}^2 : AB = \overline{CX}^2 : AX.$$

But
$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AB \cdot AP,$$

and
$$\overline{CX}^2 = \overline{AC}^2 + \overline{AX}^2 - 2\,AX \cdot AP.$$

$$\therefore \frac{\overline{AC}^2 + \overline{AB}^2 - 2\,AB \cdot AP}{AB} = \frac{\overline{AC}^2 + \overline{AX}^2 - 2\,AX \cdot AP}{AX},$$

or
$$\frac{\overline{AC}^2}{AB} + AB - 2\,AP = \frac{\overline{AC}^2}{AX} + AX - 2\,AP.$$

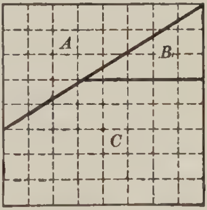
$$\therefore \frac{\overline{AC}^2}{AB} - AX = \frac{\overline{AC}^2}{AX} - AB,$$

or
$$\frac{\overline{AC}^2 - AB \cdot AX}{AB} = \frac{\overline{AC}^2 - AB \cdot AX}{AX}.$$

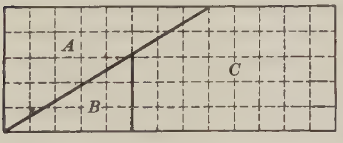
$$\therefore AB = AX.$$

5. To show geometrically that $1 = 0$.

Take a square that is 8 units on a side, and cut it into three parts, A, B, C, as shown in the right-hand figure. Fit these parts together as in the left-hand figure.



Now the square is 8 units on a side, and therefore contains 8×8, or 64, small squares, while the rectangle is 13 units long and 5 units high, and therefore contains 5×13, or 65, small squares. But the two figures are each made up of $A + B + C$ (Ax. 11), and therefore are equal (Ax. 8).

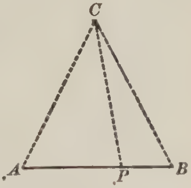


$\therefore$ $65 = 64$, and by subtracting 64 we have $1 = 0$ (Ax. 2).

6. To prove that any point on a line bisects it.

Take any point P on AB.

On AB construct an isosceles $\triangle ABC$, having $AC = BC$; and draw PC.



Then in $\triangle$ APC and PBC, we have

$\angle A = \angle B$, § 74

$AC = BC$, Const.

and $PC = PC$. Iden.

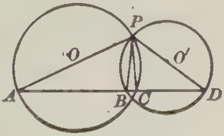
Three independent parts (that is, not merely the three angles) of one triangle are respectively equal to three parts of the other, and the triangles are congruent; therefore $AP = BP$ (§ 67).

7. To prove that it is possible to let fall two perpendiculars to a line from an external point.

Take two intersecting $\odot$ with centers O and O'.

Let one point of intersection be P, and draw the diameters PA and PD.

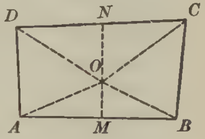
Draw AD cutting the circumferences at B and C. Then draw PB and PC.



Since $\angle PCA$ is inscribed in a semicircle, it is a right angle. In the same way, since $\angle DBP$ is inscribed in a semicircle, it also is a right angle.

$\therefore$ PB and PC are both $\perp$ to AD.

8. To prove that if two opposite sides of a quadrilateral are equal the figure is an isosceles trapezoid.



Given the quadrilateral $ABCD$, with $BC = DA$.

To prove that AB is $\parallel$ to DC.

Draw MO and NO, the $\perp$ bisectors of AB and CD, to meet at O.

If AB and DC are parallel, the proposition is already proved.

If AB and DC are not parallel, then MO and NO will meet at O, either inside or outside the figure. Let O be supposed to be inside the figure.

Draw OA, OB, OC, OD.

Then since OM is the $\perp$ bisector of AB, $\therefore OA = OB$.

Similarly $$OD = OC.$$

But DA is given equal to BC.

$\therefore \triangle AOD$ is congruent to $\triangle BOC$,

and $$\angle DOA = \angle BOC.$$

Also, rt. $\triangle$s OCN and ODN are congruent,

and $$\angle NOD = \angle CON.$$

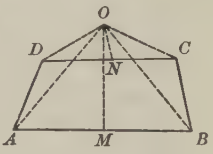
Similarly rt. $\triangle$s AMO and BMO are congruent,

and $$\angle AOM = \angle MOB.$$

$$\therefore \angle NOD + \angle DOA + \angle AOM = \angle CON + \angle BOC + \angle MOB,$$

or $$\angle NOM = \angle MON = \text{a st. } \angle.$$

Therefore the line MON is a straight line, and hence AB is $\parallel$ to DC.



If the point O is outside the quadrilateral, as in the second figure, the proof is substantially the same.

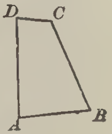
For it can be easily shown that

$$\angle DON - \angle DOA - \angle AOM$$
$$= \angle NOC - \angle BOC - \angle MOB,$$

which is possible only if

$$\angle DON = \angle DOM,$$

or if ON lies along OM.



But that the proposition is not true is evident from the third figure, in which $BC = DA$, but AB is not $\parallel$ to DC.

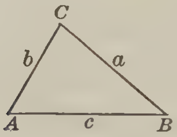
SUGGESTIONS AS TO BEGINNING DEMONSTRATIVE GEOMETRY

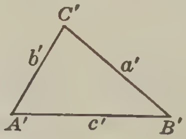
General Suggestions. When the student begins the demonstrating of propositions, whether in plane geometry or in solid geometry, it is advisable that the teacher should take a little time in which to make clear the significance of a proof and the method of attacking a new theorem.

Before assigning the proposition to be considered, the teacher will find it helpful to draw the figure on the blackboard, to state precisely what is given in certain special cases in connection with the figure, and then to ask the class to look carefully at the drawing and to state what conclusions seem to follow from what is given. There may be a number of such inferences, some of them being incorrect; but in any case the teacher should write them all on the blackboard, and the class should then examine each one to see if it seems reasonable. This is the way in which propositions are usually discovered by mathematicians, and it is the best way in which a student can be led to discover new propositions for himself and to feel that he is to a certain extent independent of the textbook.

After the inference is thought by the class to be correct, the students should be led to state the reasons for the various steps. The most desirable method at first may be that of direct questioning, — Is this statement true, and if so, for what reason? Soon, however, the students will begin to volunteer suggestions as to the argument, and they should be encouraged in this kind of independent work. A few illustrations of this method of approach will now be given.

Inferences as to Congruent Triangles. Suppose, for example, that the class is about to begin the study of congruent triangles.





The students should first consider the following questions relating to the two triangles here shown, and should draw the necessary figures to explain each answer:

1. If $\angle A = \angle A'$ and you are not sure about any of the other parts, are the triangles necessarily congruent?

If the triangles are congruent, draw two triangles having $\angle A = \angle A'$ and yet evidently not congruent. Do the same in considering the other questions given below.

2. If $\angle A = \angle A'$ and $b = b'$, are the triangles necessarily congruent?

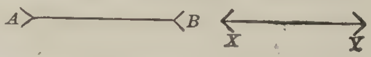
3. If $\angle A = \angle A'$, $b = b'$, and $c = c'$, are the triangles necessarily congruent?

4. If $\angle A = \angle A'$, $\angle B = \angle B'$, and $c = c'$, are the triangles necessarily congruent?

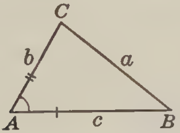
5. If $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$, are the triangles necessarily congruent?

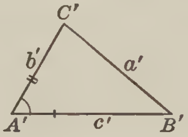
6. If $a = a'$, $b = b'$, and $c = c'$, are the triangles necessarily congruent?

Not one of the answers to the above questions may be correct. When we look at these lines we may think that they are not equal, but they are. To be certain of any inference we must find some way of proving it. Proving correct inferences or disproving incorrect ones is one of the main purposes of demonstrative geometry.



Examination of an Inference. Let us consider the third of the inferences drawn on page 454, that if $\angle A = \angle A'$, $b = b'$, and $c = c'$, the two triangles are necessarily congruent.





It aids the eye if we mark the equal corresponding parts in some such way as in the above figures. When we place the work on a blackboard we may use colored crayons, c and c' being in red, for example, and b and b' in blue, with $\angle A$ and $\angle A'$ designated by green arcs.

Teachers will see the objections to the use of colored crayons except in the case of a few propositions at the most. The student should early become familiar with the tools that he will actually use, the black lead pencil and the white crayon.

In order to prove that the two triangles are congruent, let us see if either triangle can be placed on the other so as to coincide with it. To help us see this clearly we may, if we wish, cut two triangles out of paper.

Suppose that $\triangle ABC$ is placed upon $\triangle A'B'C'$ so that the point A lies on the point A', and the side c lies along the side c'; then where does the point B lie, and why?

On what line does the side b then lie, and why?

Then where must the point C lie, and why?

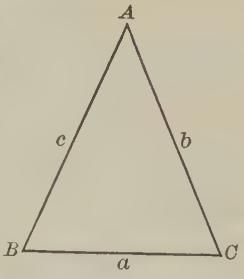
Having found where B and C lie, where does a lie?

What have we now shown with respect to $\triangle ABC$ coinciding with $\triangle A'B'C'$? Are the triangles congruent?

Complete the following statement: *Two triangles are congruent if two sides and the included angle of one are equal respectively to* · · · .

The statement and proof should now be given as in the text.

Inferences as to Isosceles Triangles. Suppose that the class is about to study the isosceles triangle. The students should consider this figure, in which $b = c$, the teacher asking questions of this kind:



1. If a is also equal to b and c, the triangle is not only isosceles, but what other name may be given to it?

2. If $\angle A$ is a right angle, the triangle is not only isosceles, but what other name may be given to it?

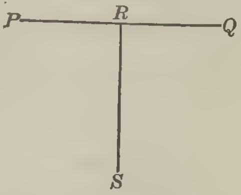
3. If $b = c$, as stated, it looks as if $\angle B$ and $\angle C$ must each be smaller than what kind of angle?

4. It looks as if there were a certain relation with respect to size between $\angle B$ and $\angle C$. What does this relation between the angles appear to be?

5. It looks as if the vertex A were directly above a certain point on BC. What point does this seem to be?

6. It looks as if a perpendicular from A to BC would divide BC into what kind of segments with respect to size?

7. The perpendicular from A to BC divides the triangle ABC into two triangles. What relation apparently exists between these two triangles?

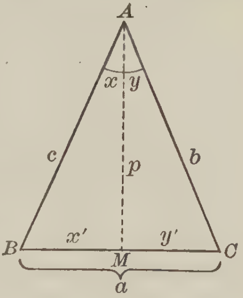


Similar questions may be asked relating, for example, to the raising or lowering of the vertex A.

No one of inferences 3–7 may be correct. When we look at this figure the line PQ seems to be about equal to the line RS, but when we measure their lengths we find that they are not equal. As already stated on page 454, we must find some way of proving or disproving our inferences before we can be certain of their truth, and this constitutes the important part of demonstrative geometry.

Further Inferences. There are other inferences that we may easily draw from a study of the isosceles triangle. Consider, for example, this figure, AM being drawn so as to bisect $\angle A$, thus making $\angle x$ equal to $\angle y$.

1. Since $\angle x = \angle y$, what seems to be the relation of x' to y'?



State to the class, if this has not already been done, that it is often convenient to use a prime (′) to designate a quantity which has some definite relation to another quantity. Also state again that it is convenient to use a dotted line to represent a line, like AM, that is an auxiliary line,—one that is drawn merely to aid us in a discussion.

2. What seems to be the relation between the two angles at M, made by the lines p and BC? Then what name can be given to each of the angles?

3. What kind of line does AM, or p, seem to be with respect to BC, or a?

4. If we draw the line p so as to bisect a instead of bisecting $\angle A$, that is, so as to make x' equal to y', what seems to be the relation as to size between $\angle x$ and $\angle y$?

5. If we draw the line p so as to make x' equal to y', what kind of line does it seem to be with respect to being oblique or perpendicular to a?

6. If a perpendicular is drawn to a at its mid-point M, do you think it will pass through A or not? What else can you infer, say with respect to $\angle A$?

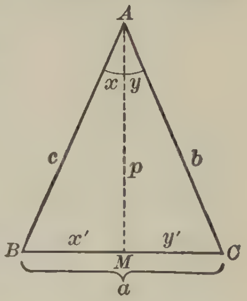
As already stated, no one of these inferences may be correct; and if we wish to be certain as to any one of them, we must prove the truth of that inference, using only definitions, axioms, postulates, or preceding propositions to assist us.

We shall now examine one of the most important of the inferences respecting the isosceles triangle.

An Inference Examined. In the fourth of the questions on page 456 the student will probably draw the inference that $\angle B = \angle C$. The members of the class should then examine this inference and see how to prove that it is correct; that is, how we can prove that

if $b = c$,

it follows that $\angle B = \angle C$.



The proposition already proved about congruent triangles stated that certain angles are equal. Tell the class that possibly we may be able to prove that $\angle B = \angle C$ if we can divide $\triangle ABC$ into two congruent triangles.

In order to use that proposition we must have two sides and the included angle of one triangle equal respectively to two sides and the included angle of another triangle; therefore, in order to get two equal angles, let us suppose, as on page 457, that AM is the bisector of $\angle A$.

Then, in $\triangle ABM$ and ACM, ask for the relation of b to c with respect to size. How is this known?

Ask for the relation of $\angle x$ to $\angle y$ with respect to size. How is this known?

Ask what line is the same in $\triangle ABM$ and ACM; that is, what line is *common* to the two triangles.

Then ask what parts of one triangle have been shown to be equal to what parts of the other triangle.

What can be said as to congruence of the triangles? How is this known?

If the triangles are congruent, what can be said as to the relation of $\angle B$ to $\angle C$?

Let the class complete the following statement:

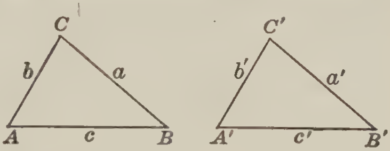
In an isosceles triangle the angles opposite the equal . . . .

The statement and proof should now be given as in the text.

Another Case of Congruence. Suppose that these two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other; that is, suppose that

$$\angle A = \angle A',$$
$$\angle B = \angle B',$$

and $c = c'$.



Ask the student to consider the general appearance of the triangles and to state his inference as to their congruence. Draw two or three other pairs of triangles, subject to similar conditions, and ask the same question.

Examination of the Inference. The students should then see whether one of the triangles can be placed on the other so as to coincide with it; in other words, each student should be certain that all the parts of one triangle fit perfectly the respective parts of the other. The teacher may proceed as follows:

Suppose that $\triangle ABC$ is placed upon $\triangle A'B'C'$ so that A lies on A' and c lies along c', C and C' lying on the same side of c'. Then where does B lie? How do you know that it lies there?

On what line does the line b then lie? How do you know that it lies there?

On what line does the line a then lie? How do you know that it lies there?

Because the point C is on both the lines a and b, at what point does it lie on the lines a' and b'?

What have you now shown with respect to the triangles?

Have you fully proved the inference about the congruence of the triangles or do you merely think from the appearance of the figures that it is probably true?

Let the class complete the following statement:

Two triangles are congruent if two angles and the included . . . .

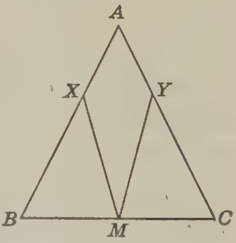
The statement and proof should now be given as in the text.

Attacking an Original Exercise. Suggestions for attacking original exercises have been given in the text, but a single illustration will probably be of service to the teacher.

Suppose that the following original is given to be proved:

Two lines drawn from the mid-point of the base of an isosceles triangle making equal angles with the base meet the equal sides at points equidistant from the vertex.

1. *Draw the figure.*



It is desirable to take as general an isosceles triangle as we can, and in particular to avoid an equilateral triangle, lest our eye should be deceived by such a special figure.

It is convenient to use M for mid-point, because it is an initial; but any other letter, say the letter P, will serve the purpose. It is well to use X and Y for the special points, or some letters not likely to be confused with A, B, and C, although this is not absolutely necessary.

The figure need not be constructed with the ruler and compasses, since this would take too much time, but it should be drawn neatly and should be accurate enough for the purpose.

2. *Write down precisely what is given, and then write down precisely what is to be proved.*

That is:

Given $AB = AC$, $BM = CM$, and $\angle XMB = \angle YMC$.

To prove that $AX = AY$.

3. *Then analyze the proposition.*

For example: I can prove that $AX = AY$ if I can prove that $BX = CY$, because I already know that $AB = AC$.

I can prove that $BX = CY$ if I can prove $\triangle MBX$ and MCY congruent.

I can prove this if I can bring it under the case of two sides and the included angle or the case of two angles and the included side.

But I can do this, for $\angle B = \angle C$, because $\triangle ABC$ is isosceles, and I also know that $\angle XMB = \angle YMC$, and $BM = CM$.

I can now reverse my reasoning and prove the theorem.

APPLICATIONS OF GEOMETRY

Purpose of Geometry. When we consider the possible applications of geometry we find that they generally belong to what is known as *intuitive geometry*. The student usually knows the facts stated in the Pythagorean Theorem before he begins to study demonstrative geometry, and he can apply the proposition to the measurement of heights, distances, and the like. It is not necessary that he should demonstrate the theorem in order to do this; neither is it necessary that he should prove the propositions about similar triangles in order to measure the height of a tree by means of certain shadows, although the question of similar triangles is involved.

The purpose of *demonstrative geometry* is not to furnish means for measurement so much as to prove the truth of the means that are already known. The essence of demonstrative geometry is the proof, and whatever takes the mind away from the proof is to be condemned unless a good reason for its existence can be shown.

Reason for Applications. In education there are many features that properly find place because they interest a student in the important thing under discussion. This is the reason why we seek for simple illustrations in connection with certain theorems in geometry. Many students are led to take a greater interest in a proposition if they feel that it relates in some way to a practical question in mechanics or in mensuration. The applications could usually be given in intuitive geometry quite as well, but a moderate use of such aids in demonstrative geometry is to be commended.

Applications of Demonstrations. Since the essential thing in this kind of geometry is the demonstration, it follows that the most valuable type of application is that which carries the demonstrations of geometry over into the problems that arise in life. Such an application makes use of the essential feature of demonstrative geometry, and the more a teacher encourages the student to use his geometric reasoning in daily life, the more valuable will be the teaching of geometry.

The teacher can give better applications of this nature than the textbook, because he can make them seem more real by referring them to real situations that arise in the school or in the life of the locality.

The following are a few types of the applications of demonstrations, but they may advantageously be worded with relation to local conditions:

1. John decides that he will buy a gun if his father gives him the money. His father does give him the money. What conclusion do you draw as to John's buying the gun?

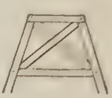
2. Kate decides to buy a dress if either her father or her mother gives her the money. Kate buys the dress. What conclusion do you draw as to the money?

3. C promises to go into business with C' if A goes into business with A' and if B goes into business with B'. If A does go into business with A', and if B goes into business with B', what follows?

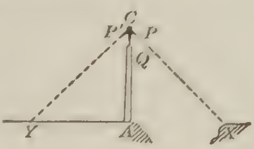
4. Suppose that it is known that a machine will run satisfactorily if three wheels properly gear into three other wheels. Suppose also that it is given that wheel a gears into wheel a', that it can be shown that wheel b gears into wheel b', and that it can then be shown that wheel c gears into wheel c'. What follows as to the running of the machine?

The reasoning is practically identical with that which the student uses in proving the first congruence theorem.

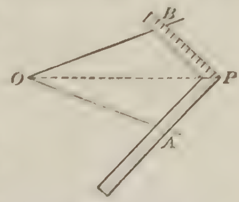
Applications of General Theorems. As already stated, applications of general theorems are introduced chiefly for the purpose of increasing the student's interest in geometry. These applications make no use of the demonstration, which is the essential part of the work, but they establish some connection between geometry and practical life. A few types of such applications are here suggested, in addition to the large number of similar problems already given in the exercises in this book.



1. In this section of a support for a heavy tank, are both cross braces necessary for rigidity? State the reason. If they are not necessary for rigidity, is there any other reason why both should be used?

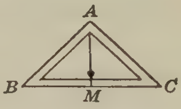


2. Wishing to measure the distance AX in this figure, a boy placed a pair of compasses QCP at the top of a post AQ so that the arm CP pointed to X. He then turned the compasses around, keeping the angle fixed, and sighted along the arm to Y. He then measured AY and thus found the distance AX. Explain the principle involved.

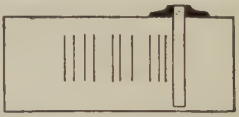


3. The following method is sometimes used for bisecting an angle by the aid of a carpenter's square: Place the square as here shown so that the edges shall pass through A and B, two points equidistant from O on the arms of the given angle AOB, and so that $AP = BP$. Draw OP and show that it bisects $\angle AOB$. It should be stated to the class that this method could be used by anyone who had never studied geometry, but that one who has studied the third congruence theorem is positive that the method gives not merely an approximate result, for he has proved that the method is absolutely exact.

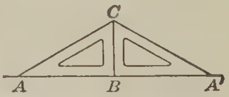
4. The ancient kind of leveling instrument here shown consists of an isosceles right triangle. When the plumb line cuts the mid-point M of the base BC, the line BC is level. State the geometric principle involved.



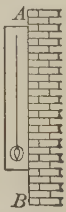
5. A draftsman draws a series of parallel lines by means of a T-square, as here shown. What is the geometric authority for stating that the lines are parallel? How would you draw a line perpendicular to these lines?



6. The accuracy of the right angle of a draftsman's triangle may be tested by first drawing a perpendicular BC to the line AA', the triangle being on the left, at ABC, and then drawing a perpendicular with the triangle on the right, at $A'BC$. State the geometric principle involved.

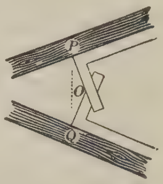


7. A bricklayer often uses the instrument here shown for determining whether a wall is vertical. When the plumb line lies along a line that is at a uniform distance from the edge AB, he knows that AB is vertical. State the geometric principle involved. State any other uses for the plumb line with which you are familiar. Are all plumb lines parallel? Consider, for example, one in London and one in Chicago.



8. In order to put in a brace joining two converging beams and making equal angles with them, a carpenter places two steel squares as here shown, so that $OP = OQ$. Show that the line PQ makes equal angles with the two beams.

9. In Ex. 8 show how a line could be drawn through the point O that would, if produced, bisect the angle that would be formed by the two beams if they were extended to the left so as to meet.



HISTORY OF GEOMETRY

Ancient Geometry. The geometry of very ancient peoples was largely the mensuration of simple areas and volumes such as is taught to children in elementary arithmetic to-day. They learned how to find the area of a rectangle, and in the oldest mathematical records that we have there is some discussion of triangles and of the volumes of solids.

Our earliest documents relating to geometry have come to us from Babylon and Egypt. Those from Babylon were written, about 2000 B.C., on small clay tablets (some of them about the size of the hand) that were afterwards baked in the sun. They show that the Babylonians of that period knew something of land measures and perhaps had advanced far enough to compute the area of a trapezoid. For the mensuration of the circle they later used, as did the early Hebrews, the value $\pi = 3$.

The first definite knowledge that we have of Egyptian mathematics comes to us from a manuscript copied on papyrus, a kind of paper used about the Mediterranean in early times. This copy was made by one Aah-mesu (the Moon-born), commonly called Ahmes, who probably flourished about 1700 B.C. The original from which he copied, written about 2300 B.C., has been lost, but the papyrus of Ahmes, written nearly four thousand years ago, is still preserved and is now in the British Museum. In this manuscript, which is devoted chiefly to fractions and to a crude algebra, is found some work on mensuration. Among the curious rules are the incorrect ones that the area of an isosceles triangle is equal to half the product of the base and one of the equal sides, and that the area of a trapezoid having bases b, b', and nonparallel sides each equal to a, is $\frac{1}{2}a(b+b')$. One noteworthy advance appears, however. Ahmes gives a rule for finding the area of a circle, substantially as follows: Multiply the square on the radius by $(\frac{16}{9})^2$, which is equivalent to taking for π the value 3.1605. Long before the time of Ahmes, however, Egypt had some knowledge of geometry, as witness the building of the Pyramids, the laying out of temples, and the digging of irrigation canals.

Early Greek Geometry. From Egypt and possibly from Babylon geometry passed to the shores of Asia Minor and Greece. The scientific study of the subject begins with Thales, one of the Seven Wise Men of the early Greek civilization. Born at Miletus about 640 B.C., he died there in 548 B.C. He founded a school of mathematics and philosophy at Miletus, known as the Ionic School. How elementary the knowledge of geometry was at that time may be understood from the fact that tradition attributes only about four propositions to Thales.

The greatest pupil of Thales, and one of the most remarkable men of antiquity, was Pythagoras. Born probably on the island of Samos, just off the coast of Asia Minor, about the year 580 B.C., Pythagoras set forth as a young man to travel. He went to Miletus and studied under Thales, probably spent several years in Egypt, and very likely went to Babylon. He then founded a school at Crotona, in Italy. In geometry he is said to have been the first to demonstrate the proposition that the square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.

Euclid. The first great textbook on geometry, and the most famous one that has ever appeared, was written by Euclid, who taught mathematics in the great university at Alexandria, Egypt, about 300 B.C. Alexandria was then practically a Greek city, having been named in honor of Alexander the Great and being ruled by the Greeks.

Euclid's work is known as the *Elements*, and, as was the case with all ancient works, the leading divisions were called "books," as is seen in the Bible and in such Latin writers as Cæsar and Vergil. This is why we speak of the various books of geometry to-day. In this work Euclid placed all the leading propositions of plane geometry as then known, and arranged them in a logical order. Most geometries of any importance since his time have been based upon this great work of Euclid, and improvements in the sequence, symbols, and wording have been made as occasion demanded.

Geometry in the East. The East did little for geometry, although contributing considerably to algebra. The first great Hindu writer was Aryabhatta, who was born 476 A.D. He gave the very close approximation for π expressed in modern notation as 3.1416. The Arabs, about the time of the Arabian Nights tales (800 A.D.), did much for mathematics, translating the Greek authors into their own language and also bringing learning from India. Indeed, it is to the Arab mathematicians of the ninth and tenth centuries that modern Europe owes its first knowledge of the *Elements* of Euclid. The Arabs, however, contributed nothing of importance to geometry.

Geometry Introduced into Europe. Euclid was translated from the Arabic into Latin in the twelfth century, Greek manuscripts not being then at hand, or being neglected because of ignorance of the language. The leading translators were Athelhard of Bath (1120), an English monk who had learned Arabic in Spain or in Egypt; Gherardo of Cremona, an Italian monk of the twelfth century; and Johannes Campanus (about 1250), chaplain to Pope Urban IV.

In the Middle Ages in Europe nothing worthy of note was added to the geometry of the Greeks. The first Latin edition of Euclid's *Elements* was printed in 1482, and the first English edition in 1570.

Solid Geometry. We have thus far spoken of the history of plane geometry. Since the Greeks were so much more interested in the logic of the science than in the applications, they did not develop the study of solid figures to the extent that they developed plane geometry. They were familiar, however, with the theory of the mensuration of the ordinary solids which we study, and of other solids of considerable complexity. Euclid considered the parallelepiped, or, as he called it, the "parallelepipedal solid." He also treated of the prism and pyramid, his proofs with relation to these figures being substantially like those given at present. Archimedes, who lived just after Euclid, completed the theory of the measurement of the round bodies, that is, of the cylinder, cone, and sphere, all of which had already been studied with some success by the Greek philosophers. Archimedes stated that the volume of a sphere is four times that of the cone with base equal to a great circle of the sphere and with height equal to the radius of the sphere. He also stated that a cylinder with base equal to a great circle of the sphere and with height equal to the diameter of the sphere is equal to one and a half times the sphere itself. So successful was he in his work with these solids that he asked that the sphere and cylinder be engraved upon his tomb, and Cicero, nearly two centuries after his death, was able to find his burial place because of this fact.

The five regular polyhedrons first occupied attention in the school of Pythagoras. The followers of this great leader knew that these figures were inscriptible in a sphere, and knew many of the important properties relating to them. Iamblichus, a Greek writer, tells us that Hippasus, a Pythagorean, perished at sea because of his impiety in boasting that he discovered the dodecahedron, although he knew that the honor was really due to Pythagoras. In the Middle Ages considerable attention was devoted to the five regular polyhedrons because of their extensive use by astrologers.

FORMULAS

Notation. The following notation is used on this page:

a = area, apothem.
a, b, c = sides of $\triangle ABC$.
a' = projection of a.
b, b' = bases.
c = circumference.
d = diameter, diagonal.
h = height, altitude.

l = length.
m = median.
p = perimeter.
r = radius.
s = semiperimeter of $\triangle$, $\frac{1}{2}(a+b+c)$.
π = about 3.1416, or about $3\frac{1}{7}$.

Formulas for Line Values. The following are important:

Right triangle,	$a^2 + b^2 = c^2$ (§ 337).
Any triangle,	$a^2 + b^2 \pm 2\,ab' = c^2$ (§§ 341, 342).
Circle,	$c = 2\,\pi r = \pi d$ (§ 385).
Radius of circle,	$r = c \div 2\,\pi$.
Equilateral triangle,	$h = \frac{1}{2}\,b\sqrt{3}$.
Diagonal of square,	$d = b\sqrt{2}$ (§ 339).
Side of square,	$b = \sqrt{a}$.

Areas of Plane Figures. The following are important:

Rectangle,	bh (§ 320).
Square,	b^2 (§ 320).
Parallelogram,	bh (§ 322).
Triangle,	$\frac{1}{2}\,bh$ (§ 325), $\sqrt{s(s-a)(s-b)(s-c)}$.
Equilateral triangle,	$\frac{1}{4}\,b^2\sqrt{3}$.
Trapezoid,	$\frac{1}{2}\,h(b+b')$ (§ 329).
Regular polygon,	$\frac{1}{2}\,ap$ (§ 386).
Circle,	$\frac{1}{2}\,rc = \pi r^2$ (§§ 388, 389).

Areas of Solid Figures. The following are the more important formulas for the areas of solid figures:

Prism,	$l = ep$ (§ 512).
Regular pyramid,	$l = \frac{1}{2} sp$ (§ 553).
Frustum of regular pyramid,	$l = \frac{1}{2}(p + p')s$ (§ 554).
Cylinder of revolution,	$l = ac = 2\pi ra$ (§ 588).
Cone of revolution,	$l = \frac{1}{2} sc = \pi rs$ (§ 609).
Frustum of cone of revolution,	$l = \frac{1}{2}(c + c')s$ (§ 615).
Sphere,	$s = 4\pi r^2$ (§ 689).
Zone,	$s = 2\pi ra$ (§ 691).
Lune	$s = \frac{\angle A}{360} \cdot 4\pi r^2 = \frac{\angle A}{90} \cdot \pi r^2$ (§ 694).

Spherical triangle,
$s =$ lune with angle equal to $\frac{1}{2}(\angle A + \angle B + \angle C - 180°)$ (§ 695).

Spherical polygon,
$s =$ lune with angle equal to $\frac{1}{2}$ (sum of $\measuredangle s - \overline{n-2} \times 180°$) (§ 699).

Volumes. The following are the more important formulas for volumes:

Rectangular parallelepiped,	$v = lwa$ (§ 534).
Prism or cylinder,	$v = ba$ (§§ 539, 589).
Pyramid or cone,	$v = \frac{1}{3} ba$ (§§ 561, 611).
Frustum of pyramid or cone,	$v = \frac{1}{3} a(b + b' + \sqrt{bb'})$ (§§ 565, 617).
Cylinder of revolution,	$v = \pi r^2 a$ (§ 590).
Circular cone,	$v = \frac{1}{3}\pi r^2 a$ (§ 612).
Frustum of cone of revolution,	$v = \frac{1}{3}\pi a(r^2 + r'^2 + rr')$ (§ 618).
Sphere,	$v = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3$ (§ 706).
Spherical pyramid,	$v = \frac{1}{3} br$.
Spherical sector,	$v = \frac{1}{3} zr$ (§ 708).
Prismatoid,	$v = \frac{1}{6} a(b + b' + 4m)$ (§ 725).
Spherical segment,	$v = \frac{1}{2} a(\pi r_1^2 + \pi r_2^2) + \frac{1}{6}\pi a^3$ (§ 726).

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